Denotational semantics for stabiliser quantum programs

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Abstract

The stabiliser fragment of quantum theory is a foundational building block for quantum error correction and the fault-tolerant compilation of quantum programs. In this article, we develop a sound, universal and complete denotational semantics for stabiliser operations which include measurement and classical control and in which quantum error-correcting codes are first-class objects. The operations are interpreted as certain *affine relations*, offering a significantly simpler alternative to the standard operator-algebraic semantics of quantum programs.

We demonstrate the power of the resulting semantics by describing a small, proof-of-concept assembly language for stabiliser programs with fully-abstract denotational semantics.

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1 Introduction

The problem of compiling quantum algorithms into fault-tolerant hardware-level instructions is a central challenge in the design of scalable quantum systems [9, 3, 30]. To this end, quantum error-correcting codes play a central role, where stabiliser codes are the most common and well-studied quantum error correction codes [21]. For fault-tolerant compilation to scale, we need a better understanding of the compositional structure of fault-tolerance, and therefore of the stabiliser fragment. Unlike general quantum programs, stabiliser quantum programs can be simulated efficiently on a probabilistic classical computer [2]. Despite this fact, the formal denotational semantics of stabiliser quantum programs has not been thoroughly studied.

In this article, we develop a nondeterministic denotational semantics for quantum programs built from stabiliser operations, including Clifford operators, Pauli errors, Pauli measurement and classically-controlled Pauli operators. Our finely tuned denotation semantics for stabilizer quantum programs is to be contrasted with the usual, much larger denotational semantics of non-stabilizer quantum programs in terms of quantum channels. Our work draws from two lines of research: the categorical semantics of quantum programming languages and quantum computing [39, 38, 37, 26]; and the symplectic representation of pure stabiliser circuits [23, 29, 25, 14, 6]. Ultimately, these results constitue the first step towards the development of formally verified fault-tolerant quantum compilation frameworks, integrating current approaches to compilation [16, 15, 24, 32] and verification [35, 11, 27, 43, 41, 18, 31, 34].

The categorical semantics of quantum theory builds on the mathematical semantics of finite-dimensional quantum processes with measurement and classical control. These semantics can be formally stated in the language of operator algebras [7], and are built in three stages of increasing expressivity:

1. *Pure quantum mechanics* via finite-dimensional Hilbert spaces;
2. *Mixed quantum mechanics* via completely-positive maps between matrix algebras;
3. *Quantum measurements and classical control* via completely-positive maps between finite-dimensional C^*-algebras.

These increasing stages of expressivity can be restated by applying the following functorial constructions to the †-compact-closed category, FHilb, of finite-dimensional Hilbert spaces and linear maps:

$$\textit{pure}\text{ QM} \xrightarrow{\text{CPM construction [37]}} \textit{mixed}\text{ QM} \xrightarrow{\text{Splitting †-idempotents [38]}} \text{QM w/ } \textit{measurements}$$

Finite-dimensional quantum mechanics can therefore be understood in purely categorical terms, agnostic to the theory of operator algebras. This point of view is highly amenable to generalisation and specialisation: simply replace FHilb with any other †-compact-closed category, and apply these constructions to add abstract notions of mixing and measurement.

In this article, we work with †-compact-closed categories specifically tailored to the stabiliser fragment. The first semantics is obtained directly by restricting FHilb to the stabiliser fragment; whereas, the second semantics is given by the symplectic representation of stabiliser maps. Specifically, we work throughout with odd-prime-dimensional quantum systems, which ensures that the symplectic representation is well-behaved. Whilst the full symplectic semantics breaks down in even characteristic, we can nevertheless recover the theory of CSS codes in the qubit case [8, 40, 13, 28].

Outline. Section 2 begins with a review of the stabiliser formalism and its symplectic formulation. We then describe novel denotational semantics for mixing in section 3 and measurement in section 4. In each section, we develop the standard operator-theoretic semantics given by restriction, and the corresponding symplectic representation, proving their equivalence:

$$\begin{tikzcd}
\text{pure stabiliser theory} \arrow[r, leftrightarrow, "\text{[29, Chapter 9],[14],(Section 2)}"] \arrow[d, "\text{CPM construction}"'] & \text{affine Lagrangian relations} \arrow[d, "\text{CPM construction}"] \\
\text{mixed stabiliser theory} \arrow[r, leftrightarrow, "\text{Section 3}"] \arrow[d, "\text{Splitting †-idempotents}"'] & \text{affine coisotropic relations} \arrow[d, "\text{Splitting †-idempotents}"] \\
\text{stabiliser theory with measurements} \arrow[r, leftrightarrow, "\text{Section 4}"] & \text{affine relations with symplectic types}
\end{tikzcd}$$

Finally, in section 5, we define a simple imperative language for stabiliser quantum programs, including Pauli measurement and classically-controlled Pauli operators, equipped with a fully abstract denotational semantics derived from section 4.

Contributions. We present several novel contributions:

- Corollary 25: a symplectic, relational semantics for completely positive stabiliser maps;
- Theorem 42: we model stabiliser quantum measurements and classical control as affine relations augmented with a modality to represent quantum data;
- Propositions 28,43: we prove that the physically-realisable stabiliser programs, i.e. *stabiliser quantum channels*, are represented by the total relations;
- Theorem 51: we interpret a toy programming language in this relational semantics, and prove full abstraction.

We do not merely apply the CPM construction and split †-idempotents to obtain our denotational semantics. We construct finely-tuned, yet equivalent, categories of relations which offer a significantly simpler alternative to the standard operator-theoretic semantics, whilst supporting concrete computational tools, unlike the naïve categorical semantics.

Notation. Throughout, p denotes an *odd* prime, so that $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ is the field of integers modulo p. Let FHilb denote the category of finite-dimensional Hilbert spaces and linear maps. The inner product is denoted by $\langle - | = \rangle$, the outer product by $|-\rangle\langle =|$, vectors by $|\varphi\rangle$, and their Hermitian adjoints by $\langle\varphi| := |\varphi\rangle^\dagger$. Denote the internal hom of linear endomorphisms on a finite-dimensional Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H}) \cong \mathcal{H}^* \otimes \mathcal{H}$.

We assume familiarity with †-symmetric monoidal categories (†-SMCs), †-compact-closed categories (†-CCCs), as well as a basic understanding of their string diagrams. See Selinger's survey article for reference [36]. Given a monoidal category $\mathcal{C}$ with an endomorphism *group* of scalars, let $\mathrm{Proj}(\mathcal{C})$ be the quotient of $\mathcal{C}$ by invertible scalars.

2 Preliminaries: the stabiliser theory

We review the elements of the stabiliser theory, and its representation with symplectic linear algebra. Explicitly, we define two †-CCCs for the pure stabiliser theory:

1. a *concrete* †-CCC, Stab_p, given by restricting FHilb;
2. an *abstract* †-CCC, $\mathsf{AffLagRel}_{\mathbb{F}_p}$, described in terms of symplectic linear algebra.

These two †-CCCss are known to be equivalent up to nonzero scalars [29, 14]. We will take $\mathsf{AffLagRel}_{\mathbb{F}_p}$ to serve as the basis from which we build our abstract denotational semantics.

2.1 The Hilbert space picture

Consider the p-dimensional complex vector space $\mathcal{H}_p := \mathbb{C}^p$, equipped with the canonical orthonormal basis $\{|x\rangle \mid x \in \mathbb{F}_p\}$. $\mathcal{H}_p$ models the state of a quantum system called a **qupit**.

▶ **Definition 1.** *Let $\chi(x) := \exp(i2\pi x/p)$, then the **Pauli operators** on $\mathcal{H}_p$ are generated by $Z|x\rangle := \chi(x)|x\rangle$ and $X|x\rangle := |x+1\rangle$ and assemble into the qupit **Pauli group**:*

$$\mathcal{P}_p := \{\chi(y)X^xZ^z \mid x, y, z \in \mathbb{F}_p\}.$$

*The n-**qupit Pauli group** is defined to be the n-fold tensor product $\mathcal{P}_p^{\otimes n}$, so that an arbitrary Pauli operator takes the form $\chi(y)\bigotimes_{j=1}^n X_j^{x_j}Z_j^{z_j}$ for some vectors $\mathbf{x}, \mathbf{z} \in \mathbb{F}_p^n$ and $y \in \mathbb{F}_p$.*

▶ **Definition 2.** *The **Clifford group** is the unitary normaliser of $\mathcal{P}_p^{\otimes n}$:*

$$\mathcal{C}\ell_p^n := \{U \in \mathcal{U}(\mathcal{H}_p^{\otimes n}) \mid \forall P \in \mathcal{P}_p^{\otimes n}, UPU^\dagger \in \mathcal{P}_p^{\otimes n}\}.$$

▶ **Lemma 3.** *Take a maximal Abelian subgroup $S \subseteq \mathcal{P}_p^{\otimes n}$ such that $\chi(x)1_{\mathcal{H}_p}^{\otimes n} \in S$ if and only if $x = 0$. Then S determines a normalised quantum state up to a global phase $\exp(2\pi i\theta)$ where $\theta \in [0, 1)$, as the unique state $|S\rangle \in \mathcal{H}_p^{\otimes n}$ such that $s|S\rangle = |S\rangle$ for all $s \in S$. The equivalence class $[\exp(2\pi i\theta)|S\rangle]_{\theta\in[0,1)}$ is called the **stabiliser state** associated to the **stabiliser group** S.*

Consider a stabiliser group $S \subseteq \mathcal{P}_p^{\otimes n}$, then for any $C \in \mathcal{C}\ell_p^n$ and $s \in S$, we have that $CsC^\dagger \cdot C|S\rangle = Cs|S\rangle = C|S\rangle$. It follows that the stabiliser group $CSC^\dagger = \{CsC^\dagger \mid s \in S\}$ stabilises the state $|CSC^\dagger\rangle = C|S\rangle$. Clifford unitaries therefore map stabiliser states to stabiliser states, and we assemble these operations into a †-CCC:

▶ **Definition 4.** *The †-CCC* Stab_p *of* ***qupit stabiliser maps*** *is the †-compact-closed subcategory of* FHilb *generated by the qupit stabiliser states and Clifford operators as well as the scalars* $1/\sqrt{p}$ *and* $\sqrt{p}$ *under tensor product, composition and the Hermitian adjoint.*

2.2 The symplectic picture

We recall how the (pure) qupit stabiliser theory can be restated in purely symplectic terms by taking the notion of a stabiliser group, and their symplectic representation, as fundamental. The following notion will serve as the objects in the "symplectic" category $\mathsf{AffLagRel}_{\mathbb{F}_p}$:

▶ **Definition 5.** *A* ***symplectic vector space*** (V, ω_V) *is a (finite-dimensional)* $\mathbb{F}_p$*-vector space* V *equipped with a non-degenerate, alternating, bilinear form* $\omega_V : V \oplus V \to \mathbb{F}_p$*.*

One can always choose the following concrete symplectic form:

▶ **Example 6** (Standard symplectic form)**.** Given any $n \in \mathbb{N}$, $(\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$ is a symplectic vector space where $\omega_n\big((\mathbf{x}, \mathbf{z}), (\mathbf{a}, \mathbf{b})\big) := \mathbf{x} \cdot \mathbf{b} - \mathbf{z} \cdot \mathbf{a}$.

The Pauli group $\mathcal{P}_p^{\otimes n}$ is a central extension of the Abelian group $\mathbb{F}_p^n \oplus \mathbb{F}_p^n$ by $\mathbb{F}_p$ [25, § 6.2.3]. This means that there is a function which parametrises Pauli operators

$$\pi : \mathbb{F}_p \oplus \mathbb{F}_p^n \oplus \mathbb{F}_p^n \longrightarrow \mathcal{P}_p^{\otimes n} : (y, \mathbf{x}, \mathbf{z}) \longmapsto \chi(y)\chi(2^{-1}\mathbf{x} \cdot \mathbf{z}) \bigotimes_{k=1}^{n} X_k^{x_k} Z_k^{z_k}, \tag{1}$$

chosen such that $\pi(y, \mathbf{x}, \mathbf{z})\pi(c, \mathbf{a}, \mathbf{b}) = \pi(y + c + 2^{-1}\mathbf{x} \cdot \mathbf{a}, \mathbf{x} + \mathbf{a}, \mathbf{z} + \mathbf{b})$. This allows us to work with Pauli operators purely in terms of symplectic data; for example:

▶ **Lemma 7.** *Two Paulis* $\pi(y, \mathbf{x}, \mathbf{z})$ *and* $\pi(c, \mathbf{a}, \mathbf{b})$ *commute if and only if* $\omega_n((\mathbf{x}, \mathbf{z}), (\mathbf{a}, \mathbf{b})) = 0$*.*

Recall that the commutation of Paulis was needed to define stabiliser groups. Therefore, the representation π allows us to define stabiliser groups purely at the symplectic level:

▶ **Definition 8.** *Given a linear subspace* S *of a symplectic vector space* (X, ω)*, its* ***symplectic complement*** *is* $S^\omega := \{\mathbf{x} \in X \mid \forall \mathbf{s} \in S, \omega_n(\mathbf{x}, \mathbf{s}) = 0\}$*. The subspace* S *is:*
- ***isotropic*** *if* $S \subseteq S^\omega$*;*
- ***coisotropic*** *if* $S^\omega \subseteq S$*;*
- ***Lagrangian*** *if it is maximally isotropic, i.e.* $S = S^\omega$*.*

An affine subspace is isotropic / coisotropic / Lagrangian if its linear part[1] *is. By convention, empty subspaces are affine Lagrangian.*

The isotropic condition on a subspace is equivalent to the commutation of the corresponding Pauli operators. The additional Lagrangian condition imposes that these subgroups must be maximal. The affine component accounts for the phase factor $\chi(a)$.

▶ **Proposition 9** ([23])**.** *The mapping* (1) *lifts to a bijection between the set of non-empty affine Lagrangian subspaces of* $(\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$ *and the set of stabiliser subgroups of* $\mathcal{P}_p^{\otimes n}$ *(i.e., the set of stabiliser states of* n *qupits up to a phase* $\chi(a)$*.).*

Symplectic vector spaces admit a notion of basis compatible with the symplectic form:

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[1] The **linear part** of an affine subspace $A \subseteq \mathbb{F}_p^n$ is the subspace of $\mathbb{F}_p^n$ given by $\{\mathbf{x} - \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in A\}$.

▶ **Proposition 10.** *Every symplectic vector space* (X, ω) *admits a basis of the form* $\{e_j, f_j \mid 1 \geqslant j \geqslant n\}$ *for some* $n \in \mathbb{N}$ *and for which, for any* j, k,

$$\omega(e_j, e_k) = 0, \quad \omega(f_j, f_k) = 0 \quad \textit{and} \quad \omega(e_j, f_k) = \begin{cases} 1 & \textit{if} \quad j = k; \\ 0 & \textit{otherwise.} \end{cases} \tag{2}$$

Such a basis is called a ***symplectic basis****. Furthermore, if* S *is an isotropic subspace of* X*, every basis of* S *extends to a symplectic basis of* X *via a symplectic Gram-Schmidt process.*

A symplectic basis for X is equivalent to a structure-preserving isomorphism $X \cong \mathbb{F}_p^n \oplus \mathbb{F}_p^n$:

▶ **Definition 11.** *A* ***symplectomorphism*** $\varphi : (X, \omega_X) \to (Y, \omega_Y)$ *is a linear isomorphism such that for all* $\mathbf{x}, \mathbf{y} \in X$, $\omega_Y(\varphi(\mathbf{x}), \varphi(\mathbf{y})) = \omega_X(\mathbf{x}, \mathbf{y})$*. An* ***affine symplectomorphism*** *is an affine map* $X \to Y$ *whose linear part*[2] *is a symplectomorphism.*

All symplectic vector spaces (X, ω_X) are even-dimensional and symplectomorphic to a standard symplectic vector space $(\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$ for $n := \dim(X)/2$. In particular:

▶ **Proposition 12.** *There is an isomorphism between the Clifford group* $\mathcal{C}\ell_p^n$*, modulo global phase, and the group of affine symplectomorphisms on* $(\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$*.*

Affine Lagrangian relations

Affine Lagrangian subspaces are *the* fundamental notion in the symplectic representation:

1. the *graph* of an affine symplectomorphism $\varphi : (X, \omega_X) \to (Y, \omega_Y)$ is an affine Lagrangian subspace $\mathrm{Gr}(\varphi) := \{(\mathbf{x}, \varphi(\mathbf{x})) \mid x \in X \subseteq X \oplus Y\} \subseteq (X \oplus Y, -\omega_X \oplus \omega_Y)$;
2. the composition of affine symplectomorphisms, and thus Clifford operators, is compatible with the relational composition of their graphs $\mathrm{Gr}(\varphi) ; \mathrm{Gr}(\psi) = \mathrm{Gr}(\psi \circ \varphi)$;
3. the action of Clifford operators on stabiliser states is compatible with the relational composition of their corresponding affine Lagrangian subspaces.

In other words, following Weinstein [42], we adopt the motto:

Everything is an affine Lagrangian relation!

▶ **Definition 13.** *The* ***category*** $\mathsf{AffLagRel}_{\mathbb{F}_p}$ ***of affine Lagrangian relations*** *has:*
- ***objects****: symplectic vector spaces* (V, ω_V)*;*
- ***morphisms*** $(V, \omega_V) \to (W, \omega_W)$*: affine Lagrangian subspaces of* $(V \oplus W, -\omega_V \oplus \omega_W)$*;*
- ***monoidal product****: given by the direct sum* $(V \oplus W, \omega_V \oplus \omega_W)$*;*
- ***monoidal unit****: given by the trivial symplectic vector space* $I := (\mathbb{F}_p^0, 1) \cong (\mathbb{F}_p^0 \oplus \mathbb{F}_p^0, \omega_0)$*;*
- ***dagger****: given by the relational converse,* $(\mathbf{x}, \mathbf{y}) \in L^\dagger$ *if and only if* $(\mathbf{y}, \mathbf{x}) \in L$*;*
- ***†-compact structure****: given by cups* $\eta_{(V, \omega_V)} := \{(0, (\mathbf{x}, \mathbf{x}))\} : I \to (V, \omega_V)^* \oplus (V, \omega_V)$*, where* $(V, \omega_V)^* := (V, -\omega_V)$*.*

▶ **Theorem 14** ([14, 29])**.** *There is an essentially surjective and full †-compact-closed functor* $\mathrm{Rel} : \mathsf{Stab}_p \to \mathsf{AffLagRel}_{\mathbb{F}_p}$ *This restricts to a †-compact-closed equivalence when quotienting by invertible scalars:* $\mathrm{Proj}(\mathsf{Stab}_p) \simeq \mathsf{AffLagRel}_{\mathbb{F}_p}$*.*

For convenience, denote $\mathsf{AffLagRel}_{\mathbb{F}_p}(n, m) := \mathsf{AffLagRel}_{\mathbb{F}_p}((\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n), (\mathbb{F}_p^m \oplus \mathbb{F}_p^m, \omega_m))$, which via theorem 14 and equation (1) are the concrete affine Lagrangian relations representing

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[2] φ is **affine** if there is a linear map ℓ such that $\varphi(\mathbf{x}) - \varphi(\mathbf{0}) = \ell(x)$. ℓ is called the **linear part** of φ.

stabiliser maps $\mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes m}$. These concrete relations represent the basic operations of the stabiliser theory. However, the full stabiliser quantum theory is *much richer* than what we have described. For example, *stabiliser codes* are fundamental in quantum error correction, but they are not described by pure states. Similarly, the measurement and classical control of stabiliser circuits, which we have not yet discussed, are essential for error correction.

3 Stabiliser codes and mixed states

The stabiliser theory presented in Section 2 is, from the perspective of quantum computation, fundamentally limited: it is efficiently simulatable on a classical computer [20, 2]. Nevertheless, the algebraic structure of the theory—specifically, the concept of stabiliser subgroups—provides the foundation for quantum error correction (QEC). QEC leverages these elements to encode and manipulate information in a way that supports universal quantum computation, while retaining structural features which permit fault tolerance.

A **stabiliser code** is a *non-maximal* stabiliser group, i.e. an Abelian subgroup S of $\mathcal{P}_p^{\otimes n}$ such that $\chi(x) \cdot 1 \in S$ if and only if $x = 0$. Whereas a maximal stabiliser group uniquely determines a pure stabiliser state $|S\rangle$—a one-dimensional subspace of the Hilbert space $\mathcal{H}_p^{\otimes n}$—a stabiliser code determines a higher-dimensional subspace, the *codespace*:

$$\mathcal{H}_S \coloneqq \{|\varphi\rangle \in \mathcal{H}_p^{\otimes n} \mid s\,|\varphi\rangle = |\varphi\rangle \text{ for all } s \in S\} \subseteq \mathcal{H}_p^{\otimes n} \tag{3}$$

Elements of the stabiliser group impose linear constraints on the codespace. Relaxing the number of constraints therefore yields a larger subspace, while still enforcing sufficient symmetry to make it possible to detect Pauli errors.

Semantically, it is natural to view a stabiliser code not just as a subspace but as the completely mixed state on that subspace. Concretely, let $\Pi_S : \mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes n}$ denote the projection of $\mathcal{H}_p^{\otimes n}$ onto $\mathcal{H}_S$. The normalised map $\rho_S \coloneqq \Pi_S / \operatorname{Tr}(\Pi_S)$ is the *mixed state* for the uniform mixed state on $\mathcal{H}_S$. This interpretation allows stabiliser codes to be treated within the framework of mixed-state quantum mechanics, and more importantly, to obtain denotational semantics for stabiliser codes via categorical constructions of mixed state quantum theory.

In this section, we introduce two semantics for mixed stabiliser quantum mechanics by:

1. *restricting* the mixed processes to those built out stabiliser maps;
2. *generalising* the symplectic representation to affine coisotropic relations.

Because the first semantics is given by restriction, it is hard to grapple with. On the other hand, the second semantics is novel, and much more apt to reason about stabiliser codes.

3.1 Completely-positive maps between matrix algebras

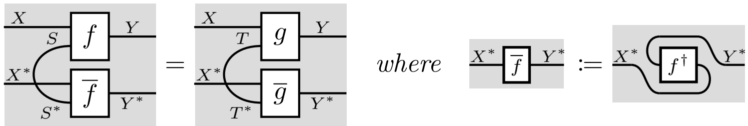
Selinger's CPM construction builds a category of mixed processes CPM(C) out of a †-CCC C by adding a notion of discarding that respects the dagger structure [37]. This plays a similar role to how the Kleisli categories Set_P and Meas_G, respectively over the power-set and Giry monads, are categorical semantics for nondeterministic and probabilistic computations:

▶ **Definition 15** ([37]). *Given a †-CCC* C*, the †-CCC* CPM(C) *has:*

- ***objects:*** *same as* C.

- ***morphisms*** $[f, S] : X \to Y$***:*** *are equivalence classes of pairs* (f, S)*, where* S *is an object of* C *and* $f : X \otimes S \to Y$ *in* C*, modulo the equivalence relation*

$$(f, S) \sim (g, T) \iff$$



- ***all other †-compact-closed structure:*** *inherited from* C*.*

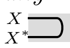
There is a canonical "doubling" functor $\iota : \mathsf{C} \to \mathrm{CPM}(\mathsf{C})$ which sends morphisms $f : X \to Y$ to $(u^R_X; f, I) : X \to Y$. This means that the CPM construction is adding a new morphism that connects both halves of this doubling. For the example of $\mathsf{C} := \mathsf{FHilb}$, this is interpreted as adding the (unnormalized) maximally mixed state:

▶ **Theorem 16** ([37, Ex. 4.21])**.** $\mathrm{CPM}(\mathsf{FHilb})$ *is equivalent to the CCC of* completely-positive *(CP) maps between matrix algebras* $\mathcal{B}(\mathcal{H})$*, for all* $\mathcal{H} \in \mathsf{FHilb}$*.*

▶ **Corollary 17.** *There is a faithful †-compact functor* $\mathrm{CPM}(\mathsf{Stab}_p) \rightarrowtail \mathrm{CPM}(\mathsf{FHilb})$*.*

The projection $\Pi_S : \mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes n}$ onto the codespace of a stabiliser code S is a state in $\mathrm{CPM}(\mathsf{Stab}_p)$. Moreover, Clifford operators C in $\mathcal{C}\ell_p^n \to \mathsf{Stab}_p \to \mathrm{CPM}(\mathsf{Stab}_p)$ act on these projectors by conjugation $C^\dagger \Pi_S C = \Pi_{C^\dagger S C}$ as expected. In particular, for a stabilizer state $|S\rangle$, we have $C^\dagger\,|S\rangle\!\langle S|\,C = \left|C^\dagger S C\right\rangle\!\!\left\langle C^\dagger S C\right|$.

To restrict physical processes, we impose the additional normalisation constraint.

▶ **Definition 18.** *Given any* $X \in \mathsf{C}$*, let* Tr_X *denote the morphism* $X \to I \in \mathrm{CPM}(\mathsf{C})$ *given by the equivalence class* *. A morphism* $[f, S] : X \to Y$ *in* $\mathrm{CPM}(\mathsf{C})$ *is* ***causal*** *if and only if* $\mathrm{Tr}_Y[f, S] = \mathrm{Tr}_X$*. Denote the symmetric monoidal subcategory of causal morphisms in* $\mathrm{CPM}(\mathsf{C})$ *by* $\mathrm{Caus}(\mathrm{CPM}(\mathsf{C}))$*.*

Concretely, the morphisms $\mathrm{Tr}_\mathcal{H}$ in $\mathrm{CPM}(\mathsf{FHilb})$, are given by the linear-algebraic trace in FHilb. In the language of operator algebras:

▶ **Corollary 19.** $\mathrm{Caus}(\mathrm{CPM}(\mathsf{FHilb}))$ *is equivalent to the SMC of completely-positive* trace-preserving *(CPTP) maps between matrix algebras* $\mathcal{B}(\mathcal{H}) \cong \mathcal{H}^* \otimes \mathcal{H}$ *for all* $\mathcal{H} \in \mathsf{FHilb}$*.*

In other words, these are CP maps which preserve the trace norm, in analogy to how Markov processes preserve the L^1 norm.

▶ **Example 20.** The normalisation $\rho_S = \Pi_S / \mathrm{Tr}(\Pi_S)$ of the projector Π_S onto the code space of a stabiliser code S is completely positive and trace preserving; whereas without the normalisation factor, the projector Π_S is only trace-preserving when $\mathrm{Tr}(\Pi_S) = 1$.

3.2 Stabiliser codes as affine coisotropic relations

In this subsection, we apply the CPM construction to the †-CCC of affine Lagrangian relations, obtaining a relational semantics for $\mathrm{CPM}(\mathsf{Stab}_p)$. In particular, we show that this produces the poset-enriched †-CCC of affine *coisotropic* relations, relaxing the dimensionality requirement for Lagrangian relations.

▶ **Definition 21.** *The †-CCC* $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ *of* ***affine coisotropic relations*** *has the same structure as* $\mathsf{AffLagRel}_{\mathbb{F}_p}$*, where now the morphisms* $(V, \omega_V) \to (W, \omega_W)$ *are affine coisotropic subspaces of* $(V \oplus W, -\omega_V \oplus \omega_W)$*.*

It is well-understood that "phaseless" stabiliser codes are in bijection with isotropic subspaces [23], and hence also with coisotropic subspaces via the symplectic complement. However, once Pauli phases are introduced, this second bijection breaks down: phased stabiliser codes correspond exactly to affine coisotropic subspaces but *not* to affine isotropic subspaces. Thus, affine coisotropic relations are the correct algebraic setting for the stabiliser theory with non-maximal stabiliser groups.

▶ **Example 22.** The total subspace can be regarded as an affine coisotropic relation:

$$\mathrm{Im}_{(V,\omega_V)} \coloneqq \{(0,\mathbf{v}) \mid \forall \mathbf{v} \in V\} : I \to (V,\omega_V) \tag{4}$$

We use the name $\mathrm{Im}_{(V,\omega_V)}$ because postcomposition with an affine Lagrangian, or affine coisotropic relation $R : (V,\omega_v) \to (W,\omega_W)$ is identified with the set-theoretic image:

$$\mathrm{Im}_V ; R = \{(0,\mathbf{w}) \mid \exists \mathbf{v} : (\mathbf{v},\mathbf{w}) \in R\} = \mathbb{F}_p^0 \oplus \mathrm{Im}(R) \cong \mathrm{Im}(R) \tag{5}$$

Adding the image as a generator to $\mathsf{AffLagRel}_{\mathbb{F}_p}$ yields $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$:

▶ **Proposition 23.** *Every non-empty affine coisotropic subspace of $\mathbb{F}_p^n \oplus \mathbb{F}_p^n$ of dimension $n+m$ is the image of an affine Lagrangian coisometry $\mathbb{F}_p^m \oplus \mathbb{F}_p^m \to \mathbb{F}_p^n \oplus \mathbb{F}_p^n$.*

Proof. We prove the claim for linear Lagrangian coisometries and coisotropic linear subspaces, after which the affine generalisation follows immediately.

Let S be such a coisotropic subspace so that S^ω is isotropic. Consider a basis of S^ω which by proposition 10 extends to a symplectic basis of (V,ω). This yields a symplectomorphism $\varphi : (V,\omega) \to (\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$ that takes this symplectic basis of V to the standard basis, and such that $\varphi(S^\omega) = \{(\mathbf{x}, \mathbf{0}_{2n-k}) \mid \mathbf{x} \in \mathbb{F}_p^k\}$. Note that the subspace $D \coloneqq \{(\mathbf{x}, \mathbf{0}_k) \mid x \in \mathbb{F}_p^k\}$ is Lagrangian in $(\mathbb{F}_p^k \oplus \mathbb{F}_p^k, \omega_k)$ so that $\varphi(S^\omega) = D \oplus \{\mathbf{0}_{2(n-k)}\}$. It follows that $C \coloneqq \mathrm{Gr}(\varphi); (D \oplus 1_{2(n-k)})$ is precisely a Lagrangian relation for which $\ker C = C^{\mathsf{R}}(\{\mathbf{0}_{2(n-k)}\}) = S^\omega$. Then, C is a composition of isometries, and thus an isometry. Then $S = (S^\omega)^\omega = (\ker C)^\omega = \mathrm{im}(C^{\mathsf{R}})$, i.e. S is the image of the Lagrangian coisometry C^{R}. ◀

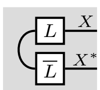
▶ **Theorem 24.** *There is a †-compact isomorphism* $\mathrm{CPM}(\mathsf{LagRel}_{\mathbb{F}_p}) \cong \mathsf{CoisotRel}_{\mathbb{F}_p}$ *sending:*

$$[f,(S,\omega_S)] : (X,\omega_S) \to (Y,\omega_Y) \quad \longmapsto \quad (1_{(X,\omega_X)} \oplus \mathrm{Im}_{(S,\omega_S)}); f : (X,\omega_S) \to (Y,\omega_Y)$$

Proof. We prove the proposition for Lagrangian and coisotropic linear relations, after which the affine extension follows immediately.

This assignment is clearly functorial and identity-on-objects, and preserves the †-compact-closed structure. Moreover, since both $\mathrm{CPM}(\mathsf{LagRel}_{\mathbb{F}_p})$ and $\mathsf{CoisotRel}_{\mathbb{F}_p}$ are compact-closed, it suffices to prove that the states in both categories are in canonical bijection. We already have surjectivity by proposition 23, so that all we need to prove is injectivity.

Given Lagrangian relations $L : S \to X$ and $M : T \to X$ such that $\mathrm{Im}(L) \neq \mathrm{Im}(M)$, then

$\mathbf{x} \in \mathrm{Im}(L)$ if and only if $\begin{bmatrix}\mathbf{x}\\ \mathbf{x}\end{bmatrix} \in$  $= \left\{ \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} \;\middle|\; \exists \mathbf{z} : \begin{array}{c}(\mathbf{z},\mathbf{x}) \in L\\ (\mathbf{z},\mathbf{y}) \in \overline{L}\end{array} \right\}.$ (6)

But by assumption there is some $\mathbf{x}$ such that $\mathbf{x} \in \mathrm{Im}(L)$ and $\mathbf{x} \notin \mathrm{Im}(M)$, and it is therefore immediate that $[L,(S,\omega_S)] \neq [M,(T,\omega_T)]$. ◀

▶ **Corollary 25.** *There is a †-compact functor* $\mathrm{Rel} : \mathrm{CPM}(\mathsf{Stab}_p) \to \mathsf{AffCoisotRel}_{\mathbb{F}_p}$*, which restricts to an equivalence when quotienting by scalars* $\mathsf{AffCoisotRel}_{\mathbb{F}_p} \simeq \mathrm{Proj}(\mathrm{CPM}(\mathsf{Stab}_p))$.

Proof. This follows immediately from the equivalence $\mathsf{AffCoisotRel}_{\mathbb{F}_p} \cong \mathrm{CPM}(\mathsf{AffLagRel}_{\mathbb{F}_p}) \simeq \mathrm{CPM}(\mathrm{Proj}(\mathsf{Stab}_p))$ and observing that, in the case of Stab_p, we obtain the same category if we quotient by scalars before or after applying the CPM construction. ◀

To include mixed states and stabiliser codes in our semantics, we update our motto:

Everything is an affine <u>coisotropic</u> relation!

▶ **Example 26.** The following quantum channels are represented in $\mathrm{Rel}(\mathrm{CPM}(\mathsf{Stab}_p))$:

- the *maximally mixed state* by $\mathrm{Im}_{(V,\omega_V)}$;
- the *quantum trace* by its relational converse $\mathrm{Im}^{\dagger}_{(V,\omega_V)}$;
- the *completely depolarising channel* by $\mathrm{Im}^{\dagger}_{(V,\omega_V)} ; \mathrm{Im}_{(V,\omega_V)}$;
- the *Z-flip channel* by $\mathscr{E}_X := \{((z,x),(z',x)) \mid x, z, z' \in \mathbb{F}_p\}$.

▶ **Definition 27.** *A relation $R : X \to Y$ with converse $R^{\dagger}$ is **total** when $\mathrm{Im}(R^{\dagger}) = X$. Given a category C of relations, let $\mathrm{Total}(\mathsf{C})$ denote the subcategory of total maps.*

By restricting the mixed stabiliser theory to the trace-preserving maps (the physical processes), the functor $\mathrm{CPM}(\mathsf{Stab}_p) \to \mathsf{AffCoisotRel}_{\mathbb{F}_p}$ restricts to an equivalence on the nose, without quotienting by scalars:

▶ **Proposition 28.** *The functor $\mathrm{Rel} : \mathrm{CPM}(\mathsf{Stab}_p) \to \mathsf{AffCoisotRel}_{\mathbb{F}_p}$ restricts to a symmetric monoidal equivalence $\mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p)) \simeq \mathrm{Total}(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Total}(\mathsf{AffCoisotRel}_{\mathbb{F}_p}) \arrow[rr, tail] \arrow[d, phantom, "\wr"] & & \mathsf{AffCoisotRel}_{\mathbb{F}_p} \arrow[d, phantom, "\wr"] \\
\mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p)) \arrow[r, tail] \arrow[d, tail] & \mathrm{CPM}(\mathsf{Stab}_p) \arrow[r, two heads] \arrow[d, tail] & \mathrm{Proj}(\mathrm{CPM}(\mathsf{Stab}_p)) \arrow[d, tail] \\
\begin{matrix} \textit{CPTP maps between} \\ \textit{matrix algebras} \end{matrix} \arrow[r, tail] & \begin{matrix} \textit{CP maps between} \\ \textit{matrix algebras} \end{matrix} \arrow[r, two heads] & \mathrm{Proj}\begin{pmatrix} \textit{CP maps between} \\ \textit{matrix algebras} \end{pmatrix}
\end{tikzcd}$$

Proof. It is immediate that $\mathrm{Rel} : \mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p)) \to \mathrm{Total}(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ is an essentially surjective, full, monoidal functor, making the diagram commute. It remains to prove faithfulness. Take two maps $[f,S],[g,T] : X \to Y$ in $\mathrm{CPM}(\mathsf{Stab})$ such that $[g,T] = \lambda \cdot [f,S]$ some $\lambda \neq 0$. Then $\mathrm{Tr}_Y[g,T] = \lambda \,\mathrm{Tr}_Y[f,S] = \lambda \,\mathrm{Tr}_X$. Therefore $[g,T]$ is causal iff $\lambda = 1$ i.e. $[g,T] = [f,S]$, thus, each projective equivalence class of morphisms $\mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p))$ contains at most one representative. Therefore, the equivalence $\mathsf{AffCoisotRel}_{\mathbb{F}_p} \simeq \mathrm{Proj}(\mathrm{CPM}(\mathsf{Stab}_p))$ uniquely lifts along Proj on causal maps. ◀

3.3 Codespace and the order on projectors

Each affine coisotropic subspace $S \subseteq (\mathbb{F}_p^{2n}, \omega_n)$ induces an orthonormal projector $\Pi_S : \mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes n}$, such that $\mathrm{Rel}(\Pi_S) = S$. In case $S = \varnothing$, then $\Pi_S := 0$. Otherwise, if $S = L + \mathbf{a}$ is nonempty, then:

$$\Pi_S := \frac{1}{|L^{\omega_n}|} \sum_{\mathbf{b} \in L^{\omega_n}} \pi\big(\omega_n(\mathbf{a},\mathbf{b}), \mathbf{b}\big).$$

Given a non-empty affine coisotropic subspace $S = L + \mathbf{a} \subseteq (\mathbb{F}_p^{2n}, \omega_n)$, we think of L^{ω_n} as the **codespace** associated to S, where Π_S is the orthogonal projector onto this codespace. Because the Pauli group $\mathcal{P}_p^{\otimes n}$ is a nice unitary error basis for $\mathcal{H}_p^{\otimes}$, we can interpret Pauli operators as potential **errors** which can occur. Moreover, because global phases are quotiented together in mixed quantum theory, we can represent arbitrary Pauli operators by elements $\mathbf{e} \in \mathbb{F}_p^{2n}$ as $\pi(\mathbf{0}, \mathbf{e})$. It is well understood that the fundamental theory of stabilizer quantum error correction can be formulated in purely symplectic terms:

An error $\mathbf{e} \in \mathbb{F}_p^{2n}$ *is*	*Symplectic condition*	*Projector condition*
Trivial	$\mathbf{e} \in L^{\omega_n}$	$\Pi_S \pi(\mathbf{0}, \mathbf{e}) \Pi_S = \Pi_S$
Detectable	$\mathbf{e} \notin L$	$\Pi_S \pi(\mathbf{0}, \mathbf{e}) \Pi_S = 0$
Undetectable, nontrivial	$\mathbf{e} \in L \setminus L^{\omega_n}$	$\Pi_S \pi(\mathbf{0}, \mathbf{e}) \Pi_S \neq 0,\ \Pi_S \pi(\mathbf{0}, \mathbf{e}) \Pi_S \neq \Pi_S$

Moreover, a finite set $\mathcal{E} \subseteq \mathbb{F}_p^{2n}$ of errors is **correctable** if and only if:

$$\forall \mathbf{e} \neq \mathbf{f} \in \mathcal{E} : \mathbf{f} - \mathbf{e} \notin L \quad \Longleftrightarrow \quad \forall \mathbf{e} \neq \mathbf{f} \in \mathcal{E} : \Pi_S \pi(\mathbf{0}, \mathbf{f} - \mathbf{e}) \Pi_S = 0.$$

The **code distance** $d(S) \in \mathbb{N}$ of a nonempty affine coisotropic subspace $S = L + \mathbf{a}$ is the minimal number of tensor factors on which a nontrivial undetectable Pauli acts. This is most easily understood in the symplectic picture where:

$$d(S) \coloneqq \min \left\{ \ \left| \{ i \in \{0, \ldots, n-1\} : (e_{x,i}, e_{z,i}) \neq (0,0) \} \right| \ : \ \forall \mathbf{e} = (\mathbf{e}_x, \mathbf{e}_z) \in L \setminus L^{\omega_n} \ \right\}$$

The order on nonempty affine Lagrangian subspaces, and their corresponding projectors is trivial. However, for affine coisotropic subspaces, it tells us when the corresponding projectors are more or less pure than each other: there is an inclusion of affine coisotropic subspaces $R \subseteq S$ if and only if there is an inclusion of the images of their corresponding projectors $\mathrm{Im}(\Pi_R) \subseteq \mathrm{Im}(\Pi_S)$. In other words, when the constraints imposed by R can be relaxed to S.

▶ **Example 29.** For an extremal example, the empty affine coisotropic subspace $\varnothing \subseteq (\mathbb{F}_p^{2n}, \omega_n)$ induces the projector from $\mathcal{H}_p^{\otimes n}$ onto the 0-dimensional subspace: $\mathbb{C}^0 \subseteq \mathcal{H}_p^{\otimes n}$. On the other extreme, the total affine coisotropic subspace $\mathbb{F}_p^{2n} \subseteq (\mathbb{F}_p^{2n}, \omega_n)$ induces the trivial projector from $\mathcal{H}_p^{\otimes n}$ onto itself. Therefore comparing these two subspaces in the symplectic picture we have an $\mathbb{F}_p$-affine subspace $\varnothing \subseteq \mathbb{F}_p^{2n}$; whereas in terms of projectors, we have a $\mathbb{C}$-linear subspace $\mathbb{C}^0 \subseteq \mathcal{H}_p^{\otimes n}$. The empty and the total affine coisotropic subspaces are respectively the bottom and top elements in the partial order on affine coisotropic subspaces, so that all affine coisotropic subspaces are contained between them.

This order also allows us to quantify error correction properties. For example, given two affine coisotropic subspaces $R, S \subseteq (\mathbb{F}_p^{2n}, \omega_n)$: $R \subseteq S$ if and only if $d(S) \subseteq d(R)$, so that code distance is anti-monotone with respect to this order.

Using the symplectic representation, we moreover have a quantitative notion to determine when *processes* are more or less pure than each other:

▶ **Theorem 30.** *By normalizing morphisms and then taking the order on projectors:*

- $\mathrm{CPM}(\mathsf{Stab}_p)$ *is enriched in preordered sets;*
- $\mathrm{Proj}(\mathrm{CPM}(\mathsf{Stab}_p))$ *is enriched in partially ordered sets;*
- $\mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p))$ *is enriched in partially ordered sets.*

Proof. Since $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ is compact closed, any morphism $R : (V, \omega_V) \to (W, \omega_W)$ canonically induces a state $\lfloor R \rfloor : I \to (V \oplus W, \omega_V \oplus \omega_W)$, called its *name*, which is analogous to taking the Choi matrix of a mixed quantum process. Therefore, there is an inclusion of affine coisotropic relations $R \subseteq S$ if and only if there is an inclusion of the images of their corresponding projectors $\mathrm{Im}(\Pi_{\lfloor R \rfloor}) \subseteq \mathrm{Im}(\Pi_{\lfloor S \rfloor})$.

Because the operator normalized Choi-matrices of morphisms between any two Hilbert spaces in Caus(CPM(Stab_p)) are distinct from each other, the isomorphism Rel : Caus(CPM(Stab_p)) $\cong$ Total($\mathsf{AffCoisotRel}_{\mathbb{F}_p}$) induces a partial order on the trace preserving mixed stabiliser quantum processes.

On the other hand, when working in CPM(Stab_p), two proportional, yet distinct morphisms of the same type induce the same projector, so the order fails to be anti-symmetric. ◀

4 Measurement and classical types

In the previous section, we saw how the CPM construction provides an abstract setting for:

1. general mixed state quantum mechanics when applied to FHilb;
2. more specifically, stabiliser codes when applied to $\mathsf{AffLagRel}_{\mathbb{F}_p} \simeq \mathsf{Stab}_p$.

As previously mentioned, stabiliser codes are used to detect and correct errors on faulty quantum channels: encoding quantum information redundantly as a mixed state. However, from, an operational point of view, to detect errors, one has to measure part of the code space; and to correct errors, one must apply operations to the code space conditional on the measurement outcomes.

Given an indexed orthonormal basis $B = \{|\lambda_1\rangle, \cdots, |\lambda_n\rangle\}$ for a finite-dimensional Hilbert space $\mathcal{H}$, the measurement in this basis is represented by the projector $\mathcal{E}_B : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ which sends pure states to probabilistic mixtures of pure states according to the *Born rule*:

$$\mathcal{E}_B(|\varphi\rangle\!\langle\varphi|) \coloneqq \sum_{j=1}^{n} |\lambda_j\rangle\!\langle\lambda_j|\,|\varphi\rangle\!\langle\varphi|\,|\lambda_j\rangle\!\langle\lambda_j| = \sum_{j=1}^{n} |\langle\lambda_j|\varphi\rangle|^2\,|\lambda_j\rangle\!\langle\lambda_j| \tag{7}$$

The indices of the basis are interpreted as the measurement outcomes occurring with probability $|\langle\lambda_j|\varphi\rangle|^2$. This projector $\mathcal{E}_B$ is an endomorphism on $\mathcal{H}$ in CPM(FHilb). In particular, a Pauli-X basis measurement is an endomorphism on $\mathcal{H}_p$ in CPM(Stab_p). Therefore, in some sense, $\mathcal{E}_B$ is the "classical" subobject of the "quantum" object $\mathcal{H}$ which has been measured in the basis B. By promoting these subobjects to objects, in the following subsection we obtain a categorical semantics for quantum theory with classical and quantum types; reproducing the usual setting for finite-dimensional quantum mechanics. Later, we perform an analogous construction to stabiliser circuits to obtain a fully relational semantics.

4.1 Adding classical types by splitting dagger-idempotents

We review the †-idempotent completion of a †-CCC, recalling:

▶ **Definition 31.** *A **†-idempotent** in a †-SMC is a map f such that $f^\dagger = f$ and $f;f = f$.*

In the setting of finite-dimensional, mixed quantum theory:

▶ **Example 32** ([26, Thm. 2.5], [12, Prop. 3.5])**.** The †-idempotents on $\mathcal{H}$ in CPM(FHilb) are in bijection with C^*-subalgebras of the matrix algebra $\mathcal{B}(H) \cong \mathcal{H}^* \otimes \mathcal{H}$.

The identity on $\mathcal{H}$ is an idempotent and corresponds to the trivial C^*-subalgebra $\mathcal{B}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$. On the other hand, projectors onto subspaces induced by measurement, such as measurement onto a basis $\mathcal{E}_B : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ correspond to commutative C^*-subalgebras of $\mathcal{B}(\mathcal{H})$. We promote these subobjects to objects:

▶ **Definition 33** ([38, Def. 3.13])**.** *Given a †-CCC* C*, the †-idempotent completion,* Split†(C)*, is the †-CCC with:*

- ***objects:*** *pairs (A, a) where A is an object of C and $a : A \to A$ is a †-idempotent;*
- ***morphisms:*** *$f : (A, a) \to (B, b)$ are morphisms $f : A \to B$ in C such that $a; f; b = f$;*
- ***identities:*** *$1_{(A,a)} \coloneqq a$;*
- ***rest of †-compact structure*** *given pointwise in C.*

There is a canonical embedding $\mathcal{C} \to \mathrm{Split}^\dagger(\mathcal{C})$ sending objects $A \mapsto (A, 1_A)$ and acting as the identity on morphisms. When applied to $\mathrm{CPM}(\mathsf{FHilb})$, the †-idempotent completion reproduces the standard setting for finite-dimensional quantum mechanics:

▶ **Theorem 34** ([26, Thm. 2.5], [12, Prop. 3.5])**.** $\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb}))$ *is equivalent to the CCC of completely-positive maps between finite-dimensional C^*-algebras.*

The objects of the form $(\mathcal{H}, 1_\mathcal{H})$ represent the matrix algebras, interpreted as the purely quantum systems in $\mathrm{CPM}(\mathsf{FHilb})$. On the other hand, the new objects added by †-idempotent completion correspond to non-matrix C^*-algebras, interpreted as being more classical. For the example of a quantum measurement induced by an orthonormal basis B, the object $(\mathcal{H}, \mathcal{E}_B)$ is interpreted as a classical system measured according to the basis B. The canonical map $\mathcal{E}_B : (\mathcal{H}, 1_\mathcal{H}) \to (\mathcal{H}, \mathcal{E}_B)$ is interpreted as the measurement induced by B; whereas the map $\mathcal{E}_B : (\mathcal{H}, \mathcal{E}_B) \to (\mathcal{H}, 1_\mathcal{H})$ is interpreted as the state preparation induced by B. Measurement followed by state preparation yields the quantum system projected onto the measurement basis; whereas, state preparation followed by measurement yields the identity on the classical system:

Measuring then preparing:

$$\begin{tikzcd} (\mathcal{H}, 1_\mathcal{H}) \arrow[d, tail, "\mathcal{E}_B"'] \arrow[dr, "\mathcal{E}_B"] & \\ (\mathcal{H}, \mathcal{E}_B) \arrow[r, two heads, "\mathcal{E}_B"'] & (\mathcal{H}, 1_\mathcal{H}) \end{tikzcd}$$

Preparing then measuring:

$$\begin{tikzcd} (\mathcal{H}, \mathcal{E}_B) \arrow[r, two heads, "\mathcal{E}_B"] \arrow[dr, equal, "\mathcal{E}_B"'] & (\mathcal{H}, 1_\mathcal{H}) \arrow[d, tail, "\mathcal{E}_B"] \\ & (\mathcal{H}, \mathcal{E}_B) \end{tikzcd}$$

Arbitrary completely-positive maps between C^*-algebras cannot be physically implemented. Just as in the previous section, we must impose an additional constraint:

▶ **Definition 35.** *A morphism $[f, S] : (X, x) \to (Y, y)$ in $\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{C}))$ is **causal** if and only if $\mathrm{Tr}_Y[f, S] = \mathrm{Tr}_X : (X, x) \to (I, 1_I)$. We denote $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{C})))$ the symmetric monoidal subcategory of causal morphisms in $\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{C}))$.*

In the setting of finite-dimensional quantum theory; this reproduces the usual operator algebraic setting for finite-dimensional quantum mechanics:

▶ **Corollary 36.** $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb})))$ *is equivalent to the symmetric monoidal category of completely-positive trace-preserving (CPTP) maps between finite-dimensional C^*-algebras.*

In other words, the morphisms in $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb})))$ correspond to finite-dimensional **quantum channels**, and the states correspond to **density matrices**. Importantly, the state preparation and measurement maps are quantum channels.

4.2 The stabiliser theory with affine nondeterministic classical control

In the previous subsection, we recalled how the symmetric monoidal category $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb})))$ is equivalent to the standard setting for quantum channels. That is to say, the finite-dimensional quantum circuits with measurement and classical control. In this subsection, by applying the same constructions to $\mathrm{CPM}(\mathsf{Stab}_p) \rightarrowtail \mathrm{CPM}(\mathsf{FHilb})$; we show that the canonical setting for stabilizer quantum mechanics with Pauli measurement and Pauli state preparation admits a concise, entirely relational description. To this end:

▶ **Definition 37.** *The †-compact-closed category $\mathsf{AffRel}_{\mathbb{F}_p}$ of **affine relations** has finite-dimensional $\mathbb{F}_p$-vector spaces as objects and affine subspaces as morphisms. Composition is given by relational composition, whilst the identity and compact-closed structure are given by the diagonal relation. The dagger is given by the relational converse.*

▶ **Lemma 38.** *There is a faithful †-compact-closed functor* $Q : \mathsf{AffLagRel}_{\mathbb{F}_p} \rightarrowtail \mathsf{AffRel}_{\mathbb{F}_p}$ *which forgets symplectic structure.*

Instead of forming $\mathrm{Split}^{\dagger}(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ on the nose, we can add additional affine relations to the image of $Q : \mathsf{AffCoisotRel}_{\mathbb{F}_p} \rightarrowtail \mathsf{AffRel}_{\mathbb{F}_p}$ which †-split †-idempotents:

▶ **Proposition 39.** *The †-idempotents in* $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ *†-split through the forgetful functor* $\mathsf{AffCoisotRel}_{\mathbb{F}_p} \rightarrowtail \mathsf{AffRel}_{\mathbb{F}_p}$*. In particular, Pauli-$X$ measurement splits through the relations:*

$$\mu_X := \left\{ \left(\begin{bmatrix} x \\ z \end{bmatrix}, x \right) \in \mathbb{F}_p^2 \oplus \mathbb{F}_p \right\} : Q(\mathbb{F}_p^2, \omega_2) \to \mathbb{F}_p, \;\; \eta_X := \left\{ \left(x, \begin{bmatrix} x \\ z \end{bmatrix} \right) \in \mathbb{Z}_p \oplus \mathbb{F}_p^2 \right\} : \mathbb{F}_p \to Q(\mathbb{F}_p^2, \omega_2)$$

Proof. Consider the relation in $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ corresponding to the Z-flip channel:

$$\mathscr{E}_X := \mathrm{Rel}(\mathcal{E}_X) = \left\{ \left(\begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} x \\ z' \end{bmatrix} \right) \in \mathbb{F}_p^2 \oplus \mathbb{F}_p^2 \right\} : (\mathbb{F}_p^2, \omega_2) \to (\mathbb{F}_p^2, \omega_2)$$

Any †-idempotent in $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ is affine symplectomorphic to $\mathscr{E}_X^{\oplus n} \oplus 1_{(\mathbb{F}_p^m \otimes \mathbb{F}_p^m, \omega_m)}$ for some $n, m \in \mathbb{N}$. Moreover, $Q(\mathscr{E}_X) = \mu_X ; \eta_X$ splits as $\eta_X ; \mu_X = 1_{\mathbb{F}_p}$. ◀

The process of †-splitting †-idempotents through $Q : \mathsf{AffCoisotRel}_{\mathbb{F}_p} \rightarrowtail \mathsf{AffRel}_{\mathbb{F}_p}$ adds a *quantum* modality Q to $\mathsf{AffRel}_{\mathbb{F}_p}$; imposing compatibility with the symplectic structure:

▶ **Definition 40.** *Let* $\mathsf{AffRel}^{Q}_{\mathbb{F}_p}$ *denote the †-CCC with:*

- ***Objects:*** *Generated by finite direct sums of finite dimensional symplectic vector spaces* $Q(V, \omega_V) \in Q(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$*, and finite dimensional vector spaces* $W \in \mathsf{AffRel}_{\mathbb{F}_p}$*;*
- ***Morphisms:*** *Generated by* $Q(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ *in addition to* $\mu_X : Q(\mathbb{F}_p^2, \omega_2) \to \mathbb{F}_p$ *and* $\nu_X : \mathbb{F}_p \to Q(\mathbb{F}_p^2, \omega_2)$ *under the direct sum and relational composition;*
- ***†-compact structure:*** *Given pointwise in* $Q(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$*,* $\mathsf{AffRel}_{\mathbb{F}_p}$*, where* $\mu_X^{\dagger} := \nu_X$*.*

By restricting to either class of objects, it is immediate that:

▶ **Lemma 41.** $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ *and* $\mathsf{AffRel}_{\mathbb{F}_p}$ *are full †-compact closed subcategories of* $\mathsf{AffRel}^{Q}_{\mathbb{F}_p}$*.*

Moreover, because $Q : \mathsf{AffCoisotRel}_{\mathbb{F}_p} \rightarrowtail \mathsf{AffRel}_{\mathbb{F}_p}$ is faithful, it is immediate that:

▶ **Theorem 42.** *There is a †-compact closed equivalence* $\mathrm{Split}^{\dagger}(\mathsf{AffCoisotRel}_{\mathbb{F}_p}) \simeq \mathsf{AffRel}^{Q}_{\mathbb{F}_p}$*.*

In other words, this category is obtained by glueing together the †-CCCs $\mathsf{AffLagRel}_{\mathbb{F}_p}$ and $\mathsf{AffRel}_{\mathbb{F}_p}$ along the map μ_X which projects onto the X subspace and its transpose. We interpret the symplectic objects $Q(V, \omega_V)$ as the quantum types; the objects W with no symplectic structure as the classical types; μ_X as the measurement in the Pauli-X basis; and ν_X as state preparation in the Pauli-X basis.

Finally, our moto becomes:

Everything is an affine relation, with quantum data captured by a <u>symplectic</u> modality!

which, admittedly, is not *quite* as catchy as the previous motos.

However, just as for $\mathrm{Split}^{\dagger}(\mathrm{CPM}(\mathsf{FHilb}))$; the category $\mathrm{Proj}(\mathrm{Split}^{\dagger}(\mathrm{CPM}(\mathsf{Stab}_p)) \cong \mathsf{AffRel}^{Q}_{\mathbb{F}_p}$ has morphisms which do not correspond to operations which can be physically implemented. We restrict ourselves to the completely-positive maps:

▶ **Proposition 43.** *The induced functor* $\mathrm{Rel} : \mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p)) \to \mathsf{AffRel}^Q_{\mathbb{F}_p}$ *restricts to a symmetric monoidal equivalence* $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p))) \simeq \mathrm{Total}(\mathsf{AffRel}^Q_{\mathbb{F}_p})$ *making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Total}(\mathsf{AffRel}^Q_{\mathbb{F}_p}) \arrow[rr, tail] & & \mathsf{AffRel}^Q_{\mathbb{F}_p} \\
\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p))) \arrow[u, "\wr"] \arrow[r, tail] \arrow[d, tail] & \mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p)) \arrow[r, two heads] \arrow[d, tail] & \mathrm{Proj}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p))) \arrow[u, "\wr"] \arrow[d, tail] \\
\begin{matrix}\textit{CPTP maps between} \\ \textit{f.d. } C^*\textit{-algebras}\end{matrix} \arrow[r, tail] & \begin{matrix}\textit{CP maps between} \\ \textit{f.d. } C^*\textit{-algebras}\end{matrix} \arrow[r, two heads] & \mathrm{Proj}\begin{pmatrix}\textit{CP maps between} \\ \textit{f.d. } C^*\textit{-algebras}\end{pmatrix}
\end{tikzcd}$$

Proof. This follows from essentially the same argument as for proposition 28. ◀

Note that classically-controlled Pauli operators can be represented in $\mathrm{Total}(\mathsf{AffRel}^Q_{\mathbb{F}_p})$ because they can be constructed with Clifford operators as well as Pauli state preparation and measurements. For notational convenience, from here onwards, denote the category of **stabiliser quantum channels** by $\mathsf{StabChan}_p := \mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p)))$ whose states are **stabiliser density matrices**.

Because all of the morphisms in $\mathrm{Total}(\mathsf{AffRel}^Q_{\mathbb{F}_p})$ are affine subspaces over $\mathbb{F}_p$, this means that exact equality of stabiliser quantum channels is computable in deterministic polynomial time. This is a deterministic analogue of the celebrated Gottesman-Knill theorem [2].

4.3 The stabiliser theory with arbitrary classical control

By relaxing the requirement that the relations between classical objects are affine relations we obtain the following category:

▶ **Definition 44.** *Let* $\mathsf{Rel}^Q_{\mathbb{F}_p}$ *denote the †-compact closed category given by the objects and morphisms of* $\mathsf{AffRel}^Q_{\mathbb{F}_p}$ *in addition to the* non-affine *relation between classical types:*

$$\left\{\left(\begin{bmatrix} a \\ b \end{bmatrix}, a \cdot b\right) \in \mathbb{F}_p^2 \oplus \mathbb{F}_p\right\} : \mathbb{F}_p^2 \to \mathbb{F}_p$$

Indeed, by adding the relation which multiplies classical dits, it follows immediately that:

▶ **Lemma 45.** *The morphisms from* $\mathbb{F}_p^n$ *to* $\mathbb{F}_p^m$ *in* $\mathsf{Rel}^Q_{\mathbb{F}_p}$ *are precisely set-relations between from the set* $\mathbb{F}_p^n$ *to the set* $\mathbb{F}_p^m$ *, ie. subsets of* $\mathbb{F}_p^n \oplus \mathbb{F}_p^m$*.*

Therefore, the total relations between classical objects are precisely functions between the underlying sets. These are exactly the classical corrections which can be performed deterministically. However, by relaxing the affine constraints between classical objects, the morphisms between quantum objects in $\mathsf{Rel}^Q_{\mathbb{F}_p}$ also fail in general to be affine coisotropic subspaces, or even just affine subspaces at all. In particular, this means that the symplectic representation of stabiliser circuits fails:

▶ **Theorem 46.** *There is no functor* $\mathrm{Total}(\mathsf{Rel}^Q_{\mathbb{F}_p}) \to \mathsf{StabChan}_p$ *which extends* $\mathrm{Total}(\mathsf{AffRel}^Q_{\mathbb{F}_p}) \simeq \mathsf{StabChan}_p$ *along* $\mathrm{Total}(\mathsf{AffRel}^Q_{\mathbb{F}_p}) \rightarrowtail \mathrm{Total}(\mathsf{Rel}^Q_{\mathbb{F}_p})$*.*

This is because the non-affine classical control of stabiliser codes can produce mixed states which are no longer proportional to uniform mixtures of pure states; non-affine corrections between basis elements can create mixtures of pure states with different weights. In other words, $\mathrm{Total}(\mathsf{Rel}^Q_{\mathbb{F}_p})$ can not tell us the probability of measurement outcomes.

Despite this fact, $\mathrm{Total}(\mathsf{Rel}^{Q}_{\mathbb{F}_p})$ still keeps track of when measurement outcomes are *possible*. For example, imagine we apply nonlinear corrections are applied to a stabiliser code, producing a quantum state which is not a stabiliser code. In $\mathrm{Total}(\mathsf{Rel}^{Q}_{\mathbb{F}_p})$ we still have a notion of when errors are detectable; namely when one can perform Pauli measurements such that the error changes the possible measurement outcomes.

Because only the possibility of errors being detected affects the code distance, $\mathrm{Total}(\mathsf{Rel}^{Q}_{\mathbb{F}_p})$ is a sound and complete semantics for the analysis of stabiliser quantum error correction protocols. Conveniently, the extra probabilistic information encoded in $\mathsf{StabChan}_p$ which does not affect the code distance is forgotten in $\mathrm{Total}(\mathsf{Rel}^{Q}_{\mathbb{F}_p})$. Moreover, because this extra information is forgotten, $\mathrm{Total}(\mathsf{Rel}^{Q}_{\mathbb{F}_p})$ retains the partial order enrichment, meaning that we can quantify when processes are more pure, or equivalently, more constrained than each other.

However, unlike in $\mathrm{Total}(\mathsf{AffRel}^{Q}_{\mathbb{F}_p})$, in $\mathrm{Total}(\mathsf{Rel}^{Q}_{\mathbb{F}_p})$, computing relational composition naïvely is exponential in the number of classical dits and quantum dits. This is to be contrasted with the efficient classical probabilistic simulation of stabiliser quantum mechanics given by the Gottesman-Knill theorem [2]. As a consequence, by taking the symplectic representation of stabiliser quantum circuits seriously, despite their probabilistic simulation being tractable, this suggests that the design of stabiliser quantum error correction protocols remains hard.

5 Case study: a small imperative language for stabiliser QEC

In this section, we introduce a minimal imperative language SPL (Stabiliser Programming Language) for stabiliser quantum channels. In other words, this is a language for quantum error correction, including measurements and classical control. This language is strongly inspired by the language QPL [39], but restricted to stabiliser operations and total, non-deterministic, *affine* classical operations.

We give SPL small-step operational semantics on pairs $[C|\rho]$ of terms acting on density operators as CPTP maps (similar to that of Ying [44, Section 3.2]), and a fully abstract denotational semantics in the SMC $\mathrm{Total}(\mathsf{AffRel}^{Q}_{\mathbb{F}_p})$. This case study serves as a proof-of-concept to demonstrate that our symplectic semantics can be used as the foundation of a quantum compilation stack whose target code is fault-tolerant by construction and with a denotational semantics amenable to formal verification. The purpose of SPL is to show that $\mathrm{Total}(\mathsf{AffRel}^{Q}_{\mathbb{F}_p})$ serves as a denotational semantics for stabiliser quantum programs, and is to be contrasted with more powerful, and computationally expressive languages such as Quipper [22] and Proto-Quipper [19] which are not specifically tailored to the stabiliser fragment.

5.1 Syntax

SPL has quantum and classical types $\mathsf{Ty} ::= \texttt{pit} \mid \texttt{qpit}$. The terms are generated from the following grammar with respect to some fixed, linearly ordered set Reg indexing registers:

$$c, d \ ::= \ c \,\mathbin{;}\, d \mid \textbf{init}\ \underline{x} \mid \underline{\mathbf{y}} = A{*}\underline{\mathbf{x}} \mid \textbf{disc}\ \underline{x} \mid \textbf{qinit}\ \underline{x} \mid \underline{\mathbf{x}} {*}{=} U \mid \textbf{meas}\ \underline{x} \mid \textbf{ctrl}_P\ \underline{x}\ \underline{y}\ \mid \textbf{skip}.$$

for all $n, m \in \mathbb{N}$, $U \in \mathcal{C}\ell^n_p$, $P \in \mathcal{P}^{\otimes n}_p$, $\mathbb{F}_p$-affine transformations $A : \mathbb{F}^n_p \to \mathbb{F}^m_p$, and $\underline{x}, \underline{y} \in \mathsf{Reg}$, $\underline{\mathbf{x}} \in \mathsf{Reg}^n$, $\underline{\mathbf{y}} \in \mathsf{Reg}^m$.

The term $c \,\mathbin{;}\, d$ represents the sequential composition of subterms; **init** $\underline{x}$ represents the initialisation of $\underline{x}$ as the p-ary digit 0; $\mathbf{y} = A{*}\underline{\mathbf{x}}$ applies the affine transformation A to $\underline{\mathbf{x}}$ and stores the result on $\underline{\mathbf{y}}$; **disc** $\underline{x}$ takes the trace of $\underline{x}$; **qinit** $\underline{x}$ represents initialisation of $\underline{x}$ as

$$\frac{\Gamma \vdash c \triangleright \Delta \quad \Delta \vdash d \triangleright \Sigma}{\Gamma \vdash c \fatsemi d \triangleright \Sigma} \qquad \frac{}{\Gamma \vdash \mathbf{init}\ \underline{x} \triangleright \underline{x} : \mathtt{pit}, \Gamma} \qquad \frac{}{\Gamma \vdash \mathbf{qinit}\ \underline{x} \triangleright \underline{x} : \mathtt{qpit}, \Gamma}$$

$$\frac{}{\underline{\mathbf{x}} : \mathtt{pit}^n, \Gamma \vdash \underline{\mathbf{y}} {=} A{*}\underline{\mathbf{x}} \triangleright \underline{\mathbf{x}} : \mathtt{pit}^n, \underline{\mathbf{y}} : \mathtt{pit}^m, \Gamma} \qquad \frac{}{\underline{x} : \mathtt{qpit}, \Gamma \vdash \mathbf{meas}\ \underline{x} \triangleright \underline{x} : \mathtt{pit}, \Gamma}$$

$$\frac{}{\Gamma \vdash \mathbf{skip} \triangleright \Gamma} \qquad \frac{}{\underline{x} : \mathtt{qpit}, \Gamma \vdash \mathbf{disc}\ \underline{x} \triangleright \Gamma} \qquad \frac{}{\underline{\mathbf{x}} : \mathtt{qpit}^n, \Gamma \vdash \underline{\mathbf{x}} {*}{=} U \triangleright \underline{\mathbf{x}} : \mathtt{qpit}^n, \Gamma}$$

$$\frac{}{\underline{x} : \mathtt{pit}, \underline{y} : \mathtt{qpit}, \Gamma \vdash \mathbf{ctrl}_P\ \underline{x}\ \underline{y} \triangleright \underline{x} : \mathtt{pit}, \underline{y} : \mathtt{qpit}, \Gamma}$$

Figure 1 Formation rules for SPL. $n \in \mathbb{N}^{>0}$ and $\tau \in \mathsf{Ty}$, $\underline{\mathbf{x}} : \tau^n$ is shorthand for $\{\underline{x_1} : \tau, \cdots, \underline{x_n} : \tau\}$ such that $\underline{\mathbf{x}} = (\underline{x_1}, \cdots, \underline{x_n}) \in \mathsf{Reg}^n$. New variables are always assumed to be fresh.

the qu*p*it $|0\rangle$; $\underline{\mathbf{x}} {*}{=} U$ applies the Clifford operator U on $\underline{\mathbf{x}}$; **meas** $\underline{x}$ represents the Pauli-X measurement on $\underline{x}$; $\mathbf{ctrl}_P\ \underline{x}\ \underline{y}$ applies the Pauli operator P on $\underline{y}$, classically controlled by $\underline{x}$ in the Pauli-X basis; and **skip** represents the identity.

SPL is equipped with an *environment-transforming* type system, which enforces linear usage of quantum data. Typed environments are partial functions $\Gamma : \mathsf{Reg} \to \mathsf{Ty}$ which bind registers to be either qupits or pits, and which we sometimes represent as $\{\underline{x} : \tau,\ \underline{y} : \sigma,\ \underline{z} : \mu,\ \cdots\}$ for $\underline{x}, \underline{y}, \underline{z} \cdots \in \mathsf{Reg}$ and $\tau, \sigma, \mu, \cdots \in \mathsf{Ty}$. We impose that the domain $\mathrm{dom}(\Gamma)$ of Γ, i.e. the set of bound registers $\{\underline{x}, \underline{y}, \underline{z}, \cdots\}$, is *finite*. Judgments are triples $\Gamma \vdash t \triangleright \Delta$ consisting of a term t and typed environments Γ, Δ. A judgment $\Gamma \vdash t \triangleright \Delta$ is **well-formed** if it is derivable from the formation rules given in figure 1.

$$\begin{array}{ll}
\{\underline{in} : \mathtt{qpit}\} \vdash & \\
\mathbf{qinit}\ \underline{x} \fatsemi \mathbf{qinit}\ \underline{out} \fatsemi & \texttt{\% initialize registers} \\
\underline{x} \mathrel{*}= F \fatsemi (\underline{x}, \underline{out}) \mathrel{*}= C_X \fatsemi & \texttt{\% prepare Bell pair} \\
(\underline{in}, \underline{x}) \mathrel{*}= C_X \fatsemi \underline{in} \mathrel{*}= F^{-1} \fatsemi \mathbf{meas}\ \underline{in} \fatsemi \mathbf{meas}\ \underline{x} \fatsemi & \texttt{\% Bell measurement} \\
\mathbf{ctrl}_Z\ \underline{in}\ \underline{out} \fatsemi \mathbf{ctrl}_X\ \underline{x}\ \underline{out} \fatsemi & \texttt{\% phase correction} \\
\mathbf{qinit}\ \underline{in} \fatsemi \mathbf{qinit}\ \underline{x} \fatsemi \mathbf{disc}\ \underline{in} \fatsemi \mathbf{disc}\ \underline{x} & \texttt{\% discard ancillae} \\
\triangleright \{\underline{out} : \mathtt{qpit}\} &
\end{array}$$

Figure 2 Qupit teleportation in SPL. Where F is the Fourier transform;C_X is the (quantum) controlled X gate; and Z and X are the Pauli Z and X gates. Input is given on register $\underline{in} : \mathtt{qpit}$ and output is returned on register $\underline{out} : \mathtt{qpit}$.

5.2 Operational semantics

In this subsection we define a structured operational semantics for SPL, which is strongly inspired by Ying's operational semantics for quantum programs [44, Section 3.2].

We interpret typed environments as objects in $\mathsf{StabChan}_p$:

▶ **Definition 47.** *Given a typed environment* Γ*, let* $\langle \Gamma \rangle$ *be the dependent tensor product:*

$$\langle \Gamma \rangle := \bigotimes_{\underline{x} \in \mathrm{dom}(\Gamma)} \left\{ (\mathcal{H}_p, 1_{\mathcal{H}_p}) \quad \textit{if} \quad \Gamma(\underline{x}) = \mathtt{qpit}; \quad \textit{else} \quad (\mathcal{H}_p, \mathcal{E}_X) \quad \textit{if} \quad \Gamma(\underline{x}) = \mathtt{pit} \right.$$

and $\mathcal{D}(\Gamma) := \mathsf{StabChan}_p(I, \langle \Gamma \rangle)$ *be the set of density operators on* $\langle \Gamma \rangle$*.*

To give our operational semantics, we establish notation to represent stabiliser quantum channels acting on subspaces of a larger ambient space. Take typed environments Γ, Δ and ordered subsets (lists) $\underline{\mathbf{x}} \subseteq \mathrm{dom}(\Gamma)$ and $\underline{\mathbf{y}} \subseteq \mathrm{dom}(\Delta)$, where moreover, $\mathrm{dom}(\Gamma) \setminus \underline{\mathbf{x}} = \mathrm{dom}(\Delta) \setminus \underline{\mathbf{y}}$. Given a stabiliser quantum channel $\mathcal{C} : \langle \Gamma|_{\underline{\mathbf{x}}} \rangle \to \langle \Delta|_{\underline{\mathbf{y}}} \rangle$ let $\mathcal{C}_{\underline{\mathbf{x}},\underline{\mathbf{y}}} : \langle \Gamma \rangle \to \langle \Gamma' \rangle$ be the stabiliser quantum channel acting as $\mathcal{C}$ on the subspace $\langle \Delta \rangle \subseteq \langle \Gamma \rangle$ and trivially on its orthogonal complement $\langle \Gamma \setminus \Delta \rangle \subseteq \langle \Gamma \rangle$.

▶ **Definition 48.** *A* ***configuration*** *is a pair consisting of a well-formed judgement* $\Gamma \vdash t \triangleright \Sigma$ *and a density operator* $\rho \in \mathcal{D}(\Gamma)$*, denoted* $[\Gamma \vdash t \triangleright \Sigma \mid \rho \in \mathcal{D}(\Gamma)]$*, or* $[t|\rho]$ *for short.*

The **small-step operational semantics** of SPL is defined by the following reduction rules, where the typed environments are omitted for notational convenience:

$$[\mathbf{skip} \mathbin{;} t|\rho] \rightsquigarrow [t|\rho] \quad [(\mathbf{init}\, \underline{x}) \mathbin{;} t|\rho] \rightsquigarrow [t|\rho; \iota(|0\rangle)_{\underline{x},\underline{x}}] \quad [(\underline{\mathbf{y}} = A * \underline{\mathbf{x}}) \mathbin{;} t|\rho] \rightsquigarrow [t|\rho; \iota(\mathcal{M}^A)_{\underline{\mathbf{x}},\underline{\mathbf{y}}}]$$

$$[(\mathbf{disc}\, \underline{x}) \mathbin{;} t|\rho] \rightsquigarrow [t|\rho; (\mathrm{Tr}_{\mathcal{H}_p})_{\underline{x},\underline{x}}] \quad [(\mathbf{qinit}\, \underline{x}) \mathbin{;} t|\rho] \rightsquigarrow [t|\rho; \iota(|0\rangle)_{\underline{x},\underline{x}}] \quad [(\underline{\mathbf{x}} \mathrel{*}= U) \mathbin{;} t|\rho] \rightsquigarrow [t|\rho; \iota(U)_{\underline{\mathbf{x}},\underline{\mathbf{x}}}]$$

$$[(\mathbf{meas}\, \underline{x}) \mathbin{;} t|\rho] \rightsquigarrow [t|\rho; (\mathcal{E}_X)_{\underline{x},\underline{x}}] \quad [(\mathbf{ctrl}_P\, \underline{x}\, \underline{y}) \mathbin{;} t|\rho] \rightsquigarrow [t|\rho; CP_{(\underline{x},\underline{y}),(\underline{x},\underline{y})}],$$

where

- $\iota : \mathsf{Stab}_p \to \mathrm{CPM}(\mathsf{Stab}_p)$ takes pure stabilizer maps to stabiliser quantum channels;
- $\mathcal{E}_X : (\mathcal{H}_p, 1_{\mathcal{H}_p}) \to (\mathcal{H}_p, \mathcal{E}_X)$ denotes the Pauli-X measurement;
- $\mathrm{Tr}_{\mathcal{H}_p} : (\mathcal{H}_p, 1_{\mathcal{H}_p}) \to I$ denotes the trace;
- $\mathcal{M}^A := \sum_{\mathbf{x} \in \mathbb{F}_p^m} |A\mathbf{x}\rangle\langle\mathbf{x}| : (\mathcal{H}_p, \mathcal{E}_x)^{\otimes m} \to (\mathcal{H}_p, \mathcal{E}_x)^{\otimes n}$;
- $CP : (\mathcal{H}_p, \mathcal{E}_X) \otimes (\mathcal{H}_p, 1_{\mathcal{H}_p}) \to (\mathcal{H}_p, \mathcal{E}_X) \otimes (\mathcal{H}_p, 1_{\mathcal{H}_p})$ is the classically controlled $P \in \mathcal{P}_p$.

The types of density operators can be inferred from the typed environments. For example, reduction rules for pit vs qupit initialisation produce different density operators:

$$[\Gamma \vdash (\mathbf{init}\, \underline{x}) \mathbin{;} t \triangleright \underline{x} : \mathtt{pit}, \Delta \mid \rho \in \mathcal{D}(\Gamma)] \rightsquigarrow [\underline{x} : \mathtt{pit}, \Gamma \vdash t \triangleright \Delta \mid \rho; \iota(|0\rangle)_{\underline{x},\underline{x}} \in \mathcal{D}(\underline{x} : \mathtt{pit}, \Gamma)]$$

$$[\Gamma \vdash (\mathbf{qinit}\, \underline{x}) \mathbin{;} t \triangleright \underline{x} : \mathtt{qpit}, \Delta \mid \rho \in \mathcal{D}(\Gamma)] \rightsquigarrow [\underline{x} : \mathtt{qpit}, \Gamma \vdash t \triangleright \Delta \mid \rho; \iota(|0\rangle)_{\underline{x},\underline{x}} \in \mathcal{D}(\underline{x} : \mathtt{qpit}, \Gamma)]$$

▶ **Definition 49.** *Two quantum channels are* ***observationally equivalent*** *if they produce the same measurement statistics according to the Born rule when acting on all density matrices.*

▶ **Theorem 50.** *The operational semantics* $\rightsquigarrow^*$ *for* SPL *is sound, complete, and universal for the observational equivalence of stabiliser quantum channels.*

Proof. It is straightforward to see that given any configuration $[t|\rho]$, there is a unique ρ' such that $[t|\rho] \rightsquigarrow^* [\mathbf{skip}|\rho']$. Therefore, given any well-formed judgement t, there is a unique stabiliser quantum channel $[t|-]$. The observational equivalence of well-formed judgements c and d under $\rightsquigarrow^*$ therefore amounts to equality as stabiliser quantum channels, $[c|-] = [d|-]$, and thus, equality as quantum channels. ◀

5.3 Denotational semantics

We give SPL a denotational semantics in $\mathrm{Total}(\mathsf{AffRel}^Q_{\mathbb{F}_p})$. On types, let $[\![\mathtt{pit}]\!] := \mathbb{F}_p$ and $[\![\mathtt{qpit}]\!] := Q(\mathbb{F}_p^2, \omega_2)$. Define the denotation of a typed environment to be the dependent direct sum $[\![\Gamma]\!] := \bigoplus_{\underline{x} \in \mathrm{dom}(\Gamma)} [\![\Gamma(\underline{x})]\!]$.

The denotation of well-formed judgments $\Gamma \vdash t \triangleright \Delta$ is given by the maps $\mathsf{AffRel}^Q_{\mathbb{F}_p}([\![\Gamma]\!], [\![\Delta]\!])$ defined inductively from the denotation of generating terms. As before, we need to establish notation to represent affine relations acting on a subset of the registers of the context. Take ordered subsets $\underline{\mathbf{x}} \subseteq \mathrm{dom}(\Gamma)$ and $\underline{\mathbf{y}} \subseteq \mathrm{dom}(\Delta)$, where moreover, $\mathrm{dom}(\Gamma) \setminus \underline{\mathbf{x}} = \mathrm{dom}(\Delta) \setminus \underline{\mathbf{y}}$.

Given a relation $S : [\![\Gamma|_{\underline{\mathbf{x}}}]\!] \to \left[\!\left[\Delta|_{\underline{\mathbf{y}}}\right]\!\right]$ let $S_{\underline{\mathbf{x}},\underline{\mathbf{y}}} : [\![\Gamma]\!] \to [\![\Gamma']\!]$ denote the relation acting as S on the subset $[\![\Delta]\!] \subseteq [\![\Gamma]\!]$ and trivially everywhere else $[\![\Gamma \setminus \Delta]\!] \subseteq [\![\Gamma]\!]$. The denotation of terms is defined inductively:

$$[\![c \mathbin{;} d]\!] := [\![c]\!] ; [\![d]\!] \qquad [\![\mathbf{skip}]\!] := 1_{[\![\Gamma]\!]} \qquad [\![\mathbf{init}\ \underline{x}]\!] := (\{(0,0)\} : \mathbb{F}_p^0 \to \mathbb{F}_p\})_{\varnothing,\underline{x}}$$

$$[\![\underline{\mathbf{y}} = A * \underline{\mathbf{x}}]\!] := (\mathrm{Gr}(A))_{\underline{\mathbf{x}},\underline{\mathbf{y}}} \qquad [\![\underline{\mathbf{x}} *= U]\!] := (\mathrm{Rel}(U))_{\underline{\mathbf{x}},\underline{\mathbf{x}}} \qquad [\![\mathbf{disc}\ \underline{x}]\!] := \{(x,0) \mid x \in \mathbb{F}_p^2\}_{\underline{x},\varnothing}$$

$$[\![\mathbf{meas}\ \underline{x}]\!] := (\mathrm{Gr}(\pi_1))_{\underline{x},\underline{x}} \qquad [\![\mathbf{qinit}\ \underline{x}]\!] := (\{(0,0)\} : \mathbb{F}_p^0 \to \mathbb{F}_p\}; \eta_X)_{\varnothing,\underline{x}}$$

$$[\![\mathbf{ctrl}_P\ \underline{x}\ \underline{y}\ ]\!] := \left\{ \left(\left(s, \begin{bmatrix} t \\ u \end{bmatrix}\right), \left(s, \begin{bmatrix} t + s\cdot a \\ u + s \cdot b \end{bmatrix}\right) \right) \;\middle|\; s,t,u \in \mathbb{F}_p \right\}_{\underline{x}\sqcup\underline{y},\underline{x}\sqcup\underline{y}}$$

where π_k is the direct-sum projection onto the k-th component, and $P = \pi(0,a,b) \in \mathcal{P}_p$. We have omitted the typed contexts, which are given in figure 1.

▶ **Theorem 51** (Full abstraction). *Well-formed judgements c and d are observationally equivalent if and only if $[\![c]\!] = [\![d]\!]$.*

Proof. Since all generating judgements are stabiliser, it follows from a straightforward induction that $[c|-]$ lies in the embedding of $\mathsf{StabChan}_p$ into the category of CPTP maps between finite-dimensional C^*-algebras. By construction $[\![c]\!] = \mathrm{Rel}[c|-]$, therefore the claim follows from proposition 43. ◀

By enlarging the permissible classical operations of SPL with arbitrary functions rather than affine transformations, the denotational semantics of SPL extends to $\mathrm{Total}(\mathsf{Rel}^{Q}_{\mathbb{F}_p})$. However, this is incompatible with the operational semantics in terms of stabiliser quantum channels, because it only tells us the possibility, rather than the probability of measurement outcomes.

6 Conclusion

We have developed a denotational semantics for stabiliser quantum programs which allows for the manipulation of stabiliser codes, Pauli measurements, and classical affine control. We demonstrated the power of this semantics by giving a fully abstract denotational semantics to a toy imperative stabiliser language.

We extended this denotational semantics with arbitrary classical control, representing the possible of measurement outcomes which can occur.

In the case of qubits, the affine, symplectic representation of stabiliser maps breaks down so that $\mathrm{Proj}(\mathsf{Stab}_2) \not\simeq \mathsf{AffLagRel}_{\mathbb{F}_2}$ [14]. By restricting the unitary operations to be generated by the controlled-not gate, the Pauli group and the swap gate, we obtain the maximal subcategory of qubit stabiliser maps on which the symplectic representation still holds [13, p. 156]. This is the natural setting for CSS codes [8, 40], which are widely used in QEC [1].

The language we have described in this paper is extremely primitive and low-level; despite the abstract geometric structure of its denotational semantics. In future work, we intend to develop a higher level programming language with primitives reflecting the elegant structure of the semantics. For example, the ability to natively represent graph states and graph-like operations, correctable and detectable errors, and the ability to make use of the enrichment in partially ordered sets would be very useful.

It is also future work to explore denotational semantics for stabiliser quantum programs using their graphical calculus. There is a complete ZX-calculus for affine Lagrangian relations [6], which is equivalent to the qupit ZX-calculus [5, 33] modulo scalars. This is an interesting direction for future work because the ZX-calculus has already been successful for constructing fault-tolerant quantum circuits [4], and the design and verification of QEC codes [10, 17].

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