# Flow Conditions for Continuous Variable Measurement-based Quantum Computing

*Robert Booth*, Damian Markham

February 19th, 2020

# Continuous variables: states

- "State space" $\to$ Hilbert space:

$$f \in \mathrm{L}^2(\mathbb{R}, \mathbb{C}) \iff f : \mathbb{R} \to \mathbb{C} \quad \text{s.t.} \quad \int_{x \in \mathbb{R}} |f(x)|^2 < \infty$$

$$\mathrm{Prob}[\mathsf{Q} \in E \mid f] = \int_{x \in E} |f(x)|^2$$

# Continuous variables: states

- "State space" $\to$ Hilbert space:

$$f \in \mathrm{L}^2(\mathbb{R}, \mathbb{C}) \iff f : \mathbb{R} \to \mathbb{C} \quad \text{s.t.} \quad \int_{x \in \mathbb{R}} |f(x)|^2 < \infty$$

$$\mathrm{Prob}[\mathsf{Q} \in E \mid f] = \int_{x \in E} |f(x)|^2$$

- Countable orthonormal basis:

$$f = \sum_{n \in \mathbb{N}} c_n \xi_n \quad \text{where} \quad \xi_n \in \mathrm{L}^2(\mathbb{R}, \mathbb{C})$$

## Continuous variables: transformations

- Seth Lloyd & Samuel Braunstein (1999):

# Continuous variables: transformations

- Seth Lloyd & Samuel Braunstein (1999):

$$\left\{\mathsf{F}, \exp(i\alpha \mathsf{Q}), \exp\left(i\beta \mathsf{Q}^2\right), \exp\left(i\gamma \mathsf{Q}^3\right), \exp(iw\mathsf{Q} \otimes \mathsf{Q})\right\}$$

# Continuous variables: transformations

- Seth Lloyd & Samuel Braunstein (1999):

$$\left\{\mathsf{F}, \mathsf{U}(\alpha, \beta, \gamma), \mathsf{CZ}(w)\right\}$$

# Continuous variables: transformations

- Seth Lloyd & Samuel Braunstein (1999):

$$\left\{\mathsf{F}, \mathsf{U}, \mathsf{CZ}(w)\right\}$$

# Continuous variables: transformations

- Seth Lloyd & Samuel Braunstein (1999):

$$\left\{ \mathrm{F}, \mathrm{U}, \mathrm{CZ}(w) \right\}$$

- Gates used for correction in the MBQC procedure:

$$\mathrm{Z}(s) \coloneqq \exp(-is\mathrm{P})$$
$$\mathrm{X}(s) \coloneqq \exp(is\mathrm{Q})$$

# Continuous variables: transformations

- Seth Lloyd & Samuel Braunstein (1999):

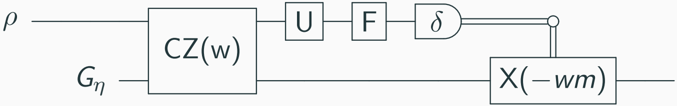
$$\left\{\mathsf{F}, \mathsf{U}, \mathsf{CZ}(w)\right\}$$

- Gates used for correction in the MBQC procedure:

$$\mathsf{Z}(s) \coloneqq \exp(-is\mathsf{P})$$
$$\mathsf{X}(s) \coloneqq \exp(is\mathsf{Q})$$

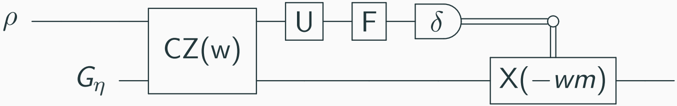
- $\Big(\text{Lie theory} \rightsquigarrow \exp : i\mathbb{R}[\mathsf{Q}, \mathsf{P}] \to \mathcal{U}(\mathcal{H})\Big)$

# Measurement-based Quantum Computing



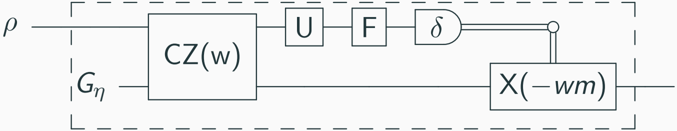
# Measurement-based Quantum Computing

- Gate teleportation



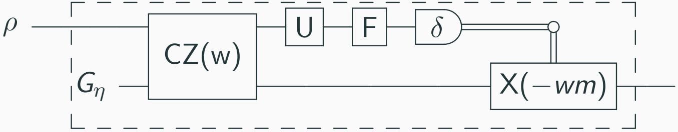
# Measurement-based Quantum Computing

- Gate teleportation



# Measurement-based Quantum Computing

- Gate teleportation




$$\updownarrow$$


$\sim \mathrm{FU}[\rho]$

- Problem 1: convergence?

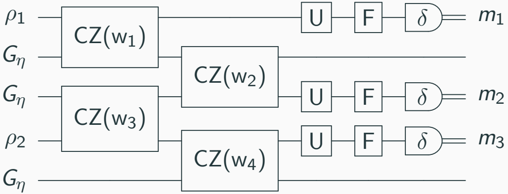
## Measurement-based Quantum Computing (cont.)

Formally,

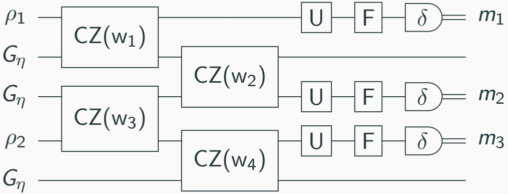
$$\lim_{\substack{\eta \to \infty \\ \delta \to 0}} \mathscr{C}_\eta^\delta[\rho] = \mathsf{S}(w)\mathsf{F}\mathsf{U}[\rho]$$

# MBQC protocols

# MBQC protocols

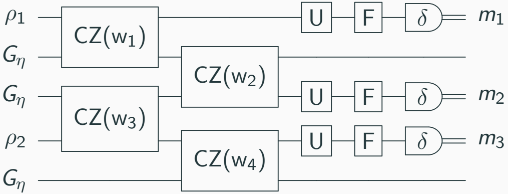


# MBQC protocols




$$\lim_{\substack{\eta\to\infty \\ \delta\to 0}} \mathscr{M}_\eta^\delta[\rho_1 \otimes \rho_2]$$

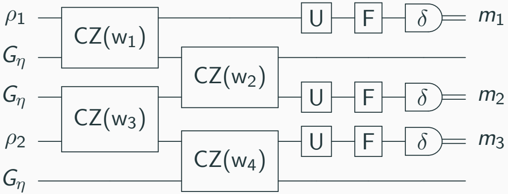
# MBQC protocols



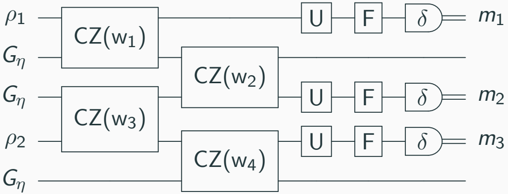

$$\lim_{\substack{\eta\to\infty\\ \delta\to 0}} \mathscr{M}_\eta^\delta[\rho_1 \otimes \rho_2]$$

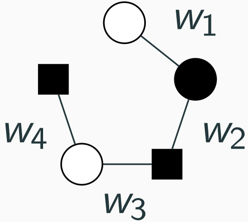
- Problem 1: convergence?
- Problem 2: runnability?

# Graph states

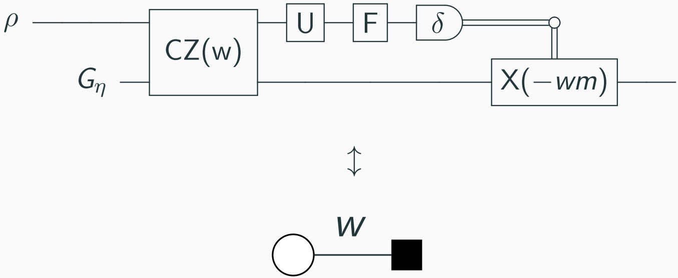


# Graph states



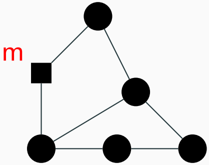



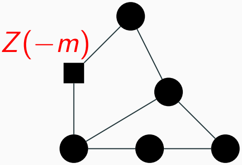
# Graph states (cont.)



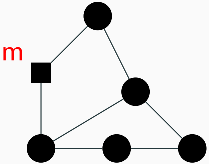
# Corrections

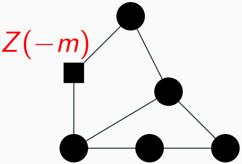
# Corrections

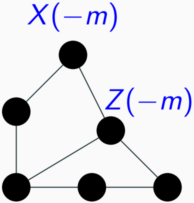
1. 

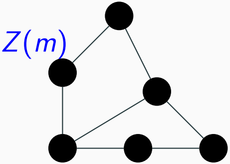
$=$


# Corrections

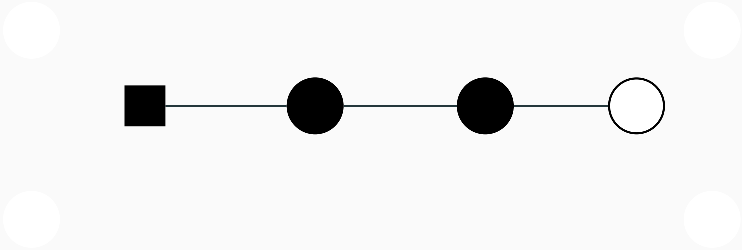
1. 

$=$


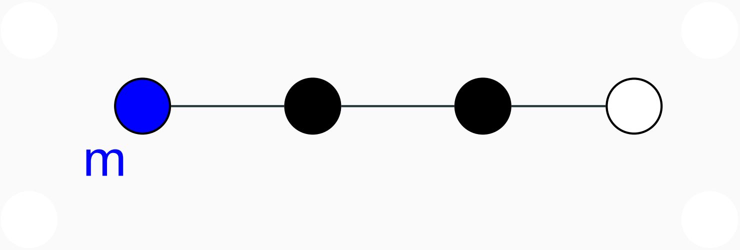

2. 

$\simeq$


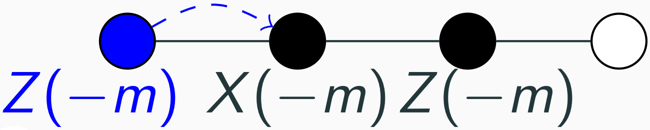
# Causal flow



# Causal flow

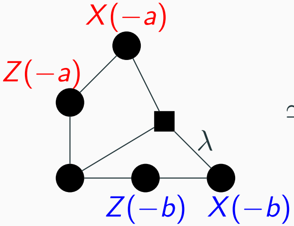


# Causal flow

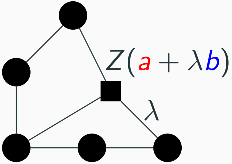


# CV-flow

# CV-flow

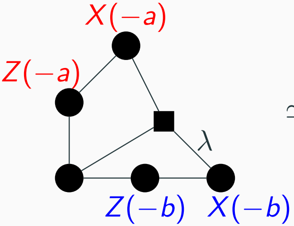
1. 


$\simeq$

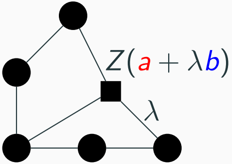


# CV-flow

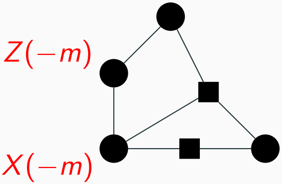
1.



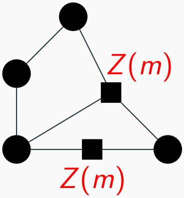

$\simeq$



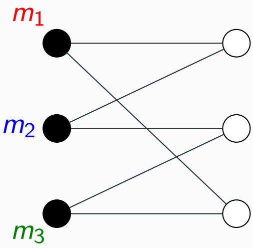

2.




$\simeq$

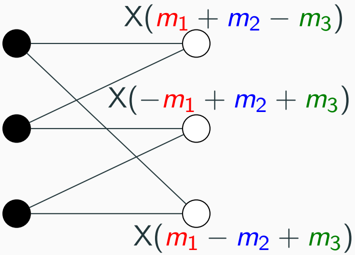


# CV-flow: an example




$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \vec{c} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

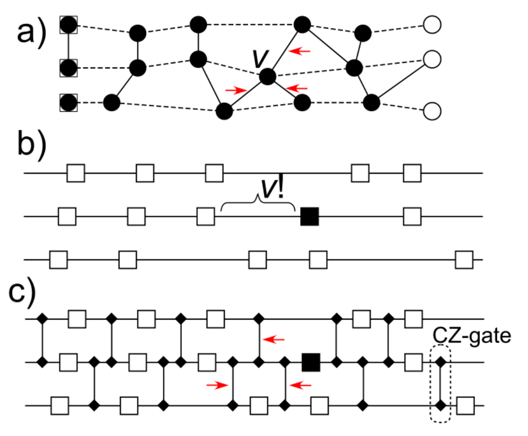
$$\vec{c} = \frac{1}{2} \begin{pmatrix} m_1 + m_2 - m_3 \\ -m_1 + m_2 + m_3 \\ m_1 - m_2 + m_3 \end{pmatrix}$$



# Recap: finding CV-flow

| | | | |
|---|---|---|---|
| Picture | Quantum Circuit | Graph State | Adjacency Matrices |
| Problem | Commute gates to make circuit runnable | Complete stabilisers | Solve linear equations |

# Extraction: causal flow




[0] Jisho Miyazaki, Michal Hajdusek, and Mio Murao. "An Analysis of the Trade-off between Spatial and Temporal Resources for Measurement-Based Quantum Computation". In: *Physical Review A 91.5* (May 2015), p. 052302

# Extraction: CV-flow

# Extraction: CV-flow

- **Lemma**: if a graph has CV-flow, then it can be split into layers, and each of these layers has a causal flow up to a *non-local* change of basis

# Extraction: CV-flow

- **Lemma**: if a graph has CV-flow, then it can be split into layers, and each of these layers has a causal flow up to a *non-local* change of basis

- **Lemma**: this change of basis doesn't get in the way of the MBQC procedure

# Extraction: CV-flow

- **Lemma**: if a graph has CV-flow, then it can be split into layers, and each of these layers has a causal flow up to a *non-local* change of basis

- **Lemma**: this change of basis doesn't get in the way of the MBQC procedure

- **Lemma**: the endpoints of paths in subsequent layers match up

## Extraction: CV-flow

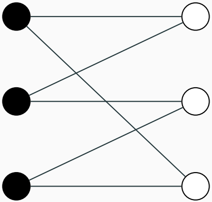
- **Lemma**: if a graph has CV-flow, then it can be split into layers, and each of these layers has a causal flow up to a *non-local* change of basis

- **Lemma**: this change of basis doesn't get in the way of the MBQC procedure

- **Lemma**: the endpoints of paths in subsequent layers match up

- **Conclusion**: the CV-flow procedure can be broken down into a sequence of Star Pattern Transformation maps with intermediate changes of basis

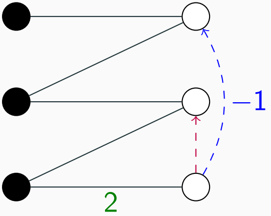
# Circuit extraction: an example



$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$C_3 \leftarrow C_3 - C_1 + C_2$$

$$A \sim \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$



# Convergence result

Formally, for any choice of input $\rho \in \mathcal{H}^{\otimes |O|}$ and any *local* unitaries,

$$\lim_{\substack{\eta \to \infty \\ \delta \to 0}} \mathscr{M}_\eta^\delta[\rho] = \prod_{i \in \mathrm{Layers}} \mathsf{T}_{\mathrm{Basis}}^{(i)} \mathsf{U}_{\mathrm{SPT}}^{(i)} \left[ \rho \right]$$

# Takeaway: results

- MBQC procedure based on CV-flow, which converges iff $|I| = |O|$.
- Explicit circuit extraction scheme for the MBQC protocol when the graph state has CV-flow.
- Poly-time algorithm for finding a CV-flow if one exists.

# Some open questions

1. A (pseudo-)necessity theorem?
2. Cleaner proofs with distributions?
3. Generalisation to other groups?
4. Counting flows and Blind Quantum Computation?