Flow Conditions for Continuous Variable Measurement-based Quantum Computing

Robert Booth, Damian Markham

June 5th, 2020

Causal flow and gflow

Causal flow and gflow

- Vincent Danos and Elham Kashefi. “Determinism in the One-Way Model”. In: *Physical Review A* 74.5 (Nov. 2006), p. 052310

Causal flow and gflow

- Vincent Danos and Elham Kashefi. "Determinism in the One-Way Model". In: *Physical Review A* 74.5 (Nov. 2006), p. 052310
- D. E. Browne et al. "Generalized Flow and Determinism in Measurement-based Quantum Computation". In: *New Journal of Physics* 9.8 (Aug. 2007), pp. 250–250

Causal flow and gflow

- Vincent Danos and Elham Kashefi. "Determinism in the One-Way Model". In: *Physical Review A* 74.5 (Nov. 2006), p. 052310
- D. E. Browne et al. "Generalized Flow and Determinism in Measurement-based Quantum Computation". In: *New Journal of Physics* 9.8 (Aug. 2007), pp. 250–250
- **miyazaki˙analysis˙2015-1**

Some motivation

- Nicolas C. Menicucci et al. "Universal Quantum Computation with Continuous-Variable Cluster States". In: *Physical Review Letters* 97.11 (Sept. 2006)
- Jun-ichi Yoshikawa et al. "Generation of One-Million-Mode Continuous-Variable Cluster State by Unlimited Time-Domain Multiplexing". In: *APL Photonics* 1.6 (Sept. 2016), p. 060801

Some motivation

- Anne Broadbent and Elham Kashefi. "Parallelizing Quantum Circuits". In: *Theoretical Computer Science* 410.26 (June 2009), pp. 2489–2510
- Ross Duncan et al. "Graph-Theoretic Simplification of Quantum Circuits with the ZX-calculus". In: *Quantum* 4 (June 2020), p. 279
- Atul Mantri et al. "Flow Ambiguity: A Path Towards Classically Driven Blind Quantum Computation". In: *Physical Review X* 7.3 (July 2017)

Continuous variables: states

- State space $\to$ Hilbert space:

$$f \in L^2(,) \iff f :\to \quad \text{s.t.} \quad \int_{x\in} |f(x)|^2 \,\mathrm{d}x < \infty$$

$$\mathrm{Prob}[Q \in E \mid f] = \int_{x\in E} |f(x)|^2 \,\mathrm{d}x$$

- Countable orthonormal basis:

$$f = \sum_{n\in\mathbb{N}} c_n \xi_n \quad \text{where} \quad \xi_n \in L^2(,)$$

Continuous variables: transformations

- Gate set:

Seth Lloyd and Samuel L. Braunstein. "Quantum Computation over Continuous Variables". In: *Physical Review Letters* 82.8 (Feb. 1999), pp. 1784–1787.

Continuous variables: transformations

- Gate set:

$$\left\{\mathsf{F}, \exp(i\alpha\mathsf{Q}), \exp\left(i\beta\mathsf{Q}^2\right), \exp\left(i\gamma\mathsf{Q}^3\right), \exp(iw\mathsf{Q}\otimes\mathsf{Q})\right\}$$

Seth Lloyd and Samuel L. Braunstein. "Quantum Computation over Continuous Variables". In: *Physical Review Letters* 82.8 (Feb. 1999), pp. 1784–1787.

Continuous variables: transformations

- Gate set:

$$\left\{\mathsf{F}, \mathsf{U}(\alpha, \beta, \gamma), \mathsf{CZ}(w)\right\}$$

Seth Lloyd and Samuel L. Braunstein. "Quantum Computation over Continuous Variables". In: *Physical Review Letters* 82.8 (Feb. 1999), pp. 1784–1787.

Continuous variables: transformations

- Gate set:

$$\left\{\mathrm{F}, \mathrm{U}, \mathrm{CZ}(w)\right\}$$

Seth Lloyd and Samuel L. Braunstein. "Quantum Computation over Continuous Variables". In: *Physical Review Letters* 82.8 (Feb. 1999), pp. 1784–1787.

Continuous variables: transformations

- Gate set:

$$\left\{\mathrm{F}, \mathrm{U}, \mathrm{CZ}(w)\right\}$$

- Gates used for correction in the MBQC procedure:

$$\mathrm{Z}(s) := \exp(-is\mathrm{P})$$
$$\mathrm{X}(s) := \exp(is\mathrm{Q})$$

Seth Lloyd and Samuel L. Braunstein. "Quantum Computation over Continuous Variables". In: *Physical Review Letters* 82.8 (Feb. 1999), pp. 1784–1787.

Continuous variables: transformations

- Gate set:

$$\left\{\mathrm{F}, \mathrm{U}, \mathrm{CZ}(w)\right\}$$

- Gates used for correction in the MBQC procedure:

$$\mathrm{Z}(s) := \exp(-is\mathrm{P})$$
$$\mathrm{X}(s) := \exp(is\mathrm{Q})$$

- $\left(\text{Lie theory} \rightsquigarrow \exp : i\mathbb{R}[\mathrm{Q}, \mathrm{P}] \to \mathcal{U}(\mathcal{H})\right)$

Seth Lloyd and Samuel L. Braunstein. "Quantum Computation over Continuous Variables". In: *Physical Review Letters* 82.8 (Feb. 1999), pp. 1784–1787.

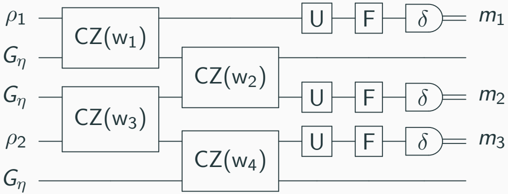
Measurement-based Quantum Computing (cont.)

Formally,

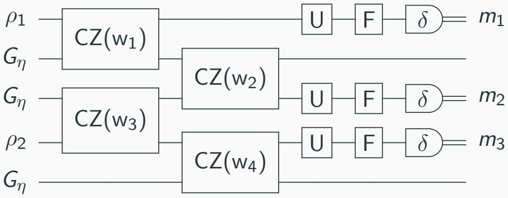
$$\lim \eta \to \infty \mathscr{G}_\eta^\delta[\rho] = S(w)FU[\rho]$$

MBQC protocols

MBQC protocols

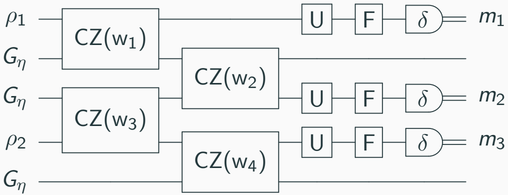


MBQC protocols



$$\lim_{\substack{\eta\to\infty \\ \delta\to 0}} \mathscr{M}_\eta^\delta[\rho_1 \otimes \rho_2]$$

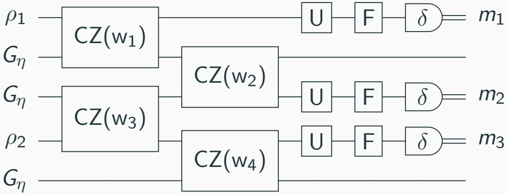
MBQC protocols



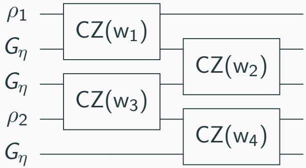
$$\lim_{\substack{\eta\to\infty \\ \delta\to 0}} \mathscr{M}_\eta^\delta[\rho_1 \otimes \rho_2]$$

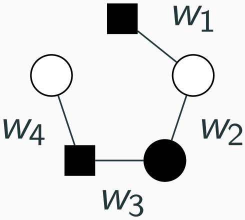
- Problem 1: convergence?
- Problem 2: runnability?

Graph states

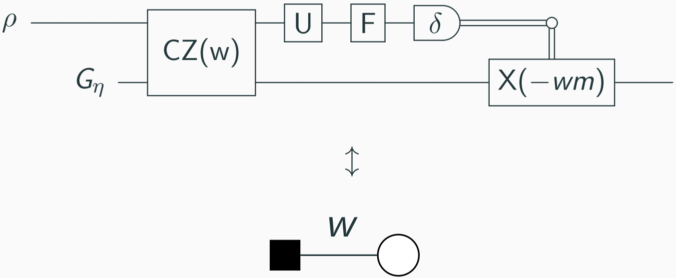


Graph states





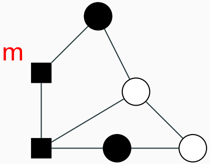
Graph states (cont.)

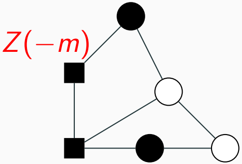


Problem I: Runnability

Corrections

Corrections

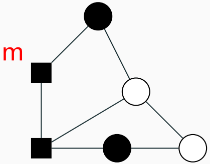
1. 

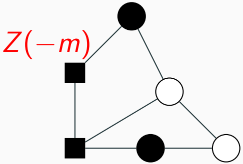
$=$ 

"error $\rightsquigarrow$ translation"

$$P_m = Z(m) P_0 Z(-m)$$

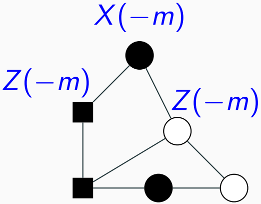
Corrections

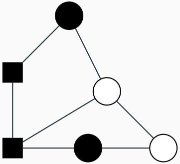
1. 

$=$ 

"error $\rightsquigarrow$ translation"

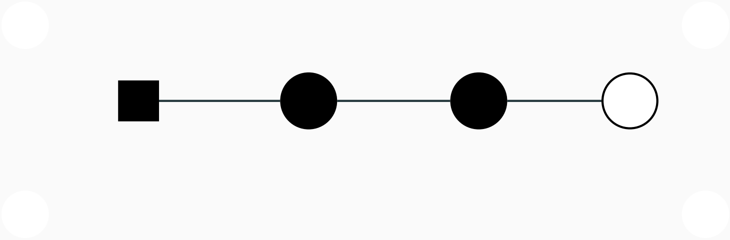
$$P_m = Z(m)P_0Z(-m)$$

2. 

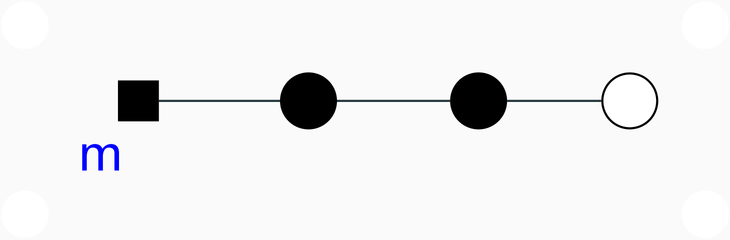
$\simeq$ 
"correction $\rightsquigarrow$ stabiliser"

$$X(m)\prod_{neighbours} Z(m)G = G$$

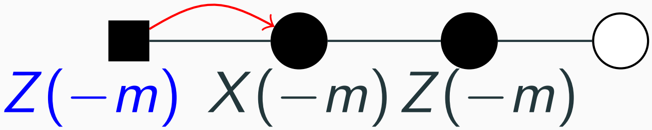
Causal flow



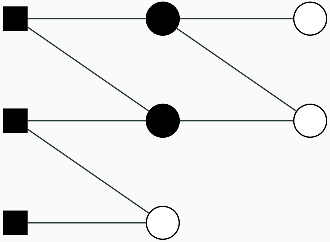
Causal flow



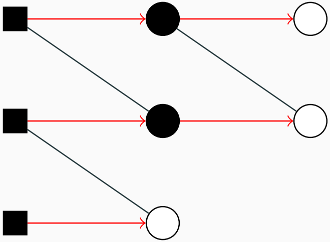
Causal flow



Causal flow



Causal flow

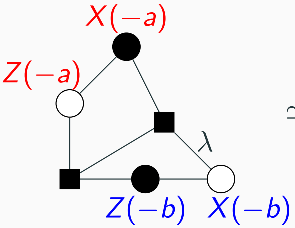


CV-flow

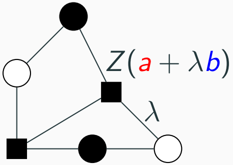
1. $Z(a+b) = Z(a) + Z(b)$

CV-flow

1. $Z(a+b) = Z(a) + Z(b)$

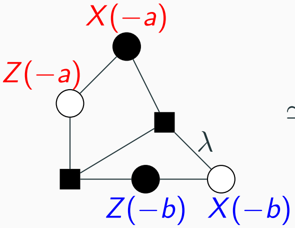
2. 

$\simeq$

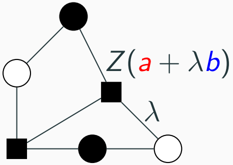


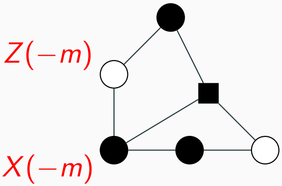
CV-flow

1. $Z(a+b) = Z(a) + Z(b)$

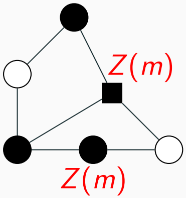
2. 

$\simeq$

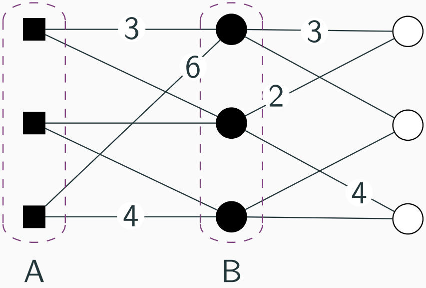


3. 

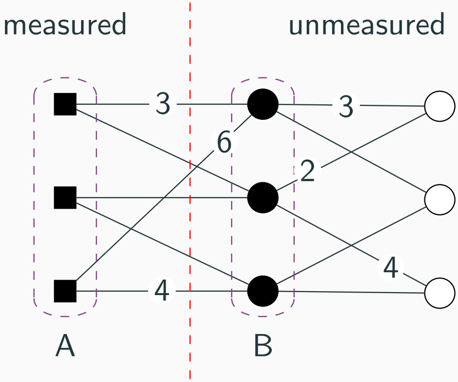
$\simeq$



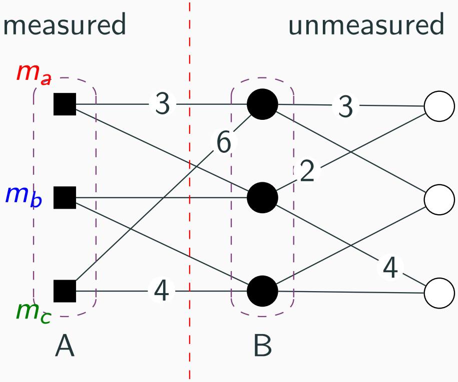
CV-flow: an example



CV-flow: an example

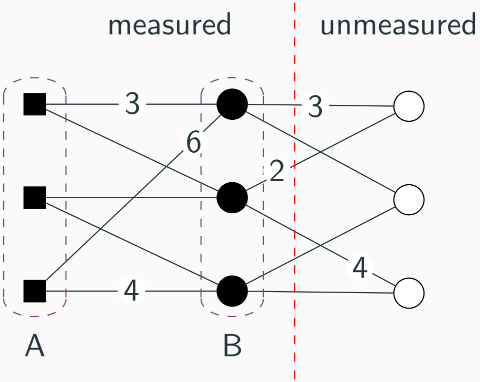


CV-flow: an example

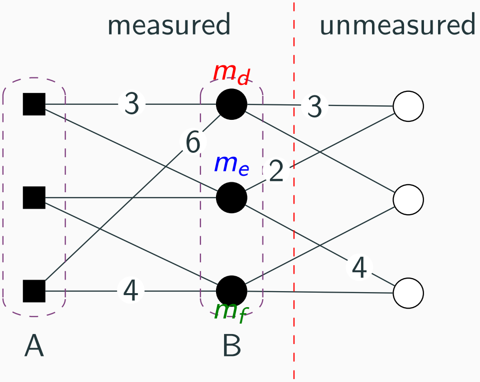


$$\begin{pmatrix} 3 & 0 & 6 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix} \vec{c_2} = \begin{pmatrix} m_a \\ m_b \\ m_c \end{pmatrix}$$

CV-flow: an example



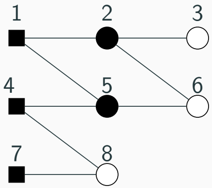
CV-flow: an example



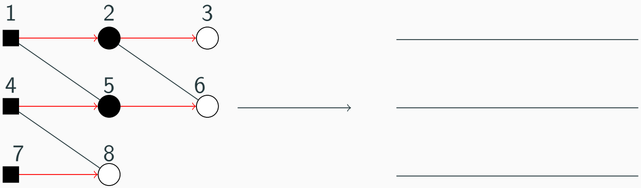
$$\begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 4 & 1 \end{pmatrix} \vec{c}_2 = \begin{pmatrix} m_d \\ m_e \\ m_f \end{pmatrix}$$

Problem II: Convergence

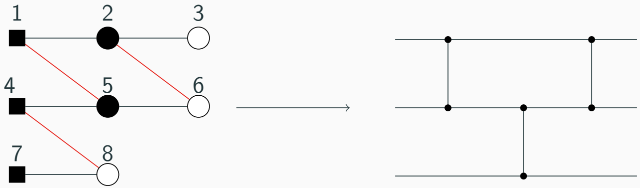
Extraction: causal flow



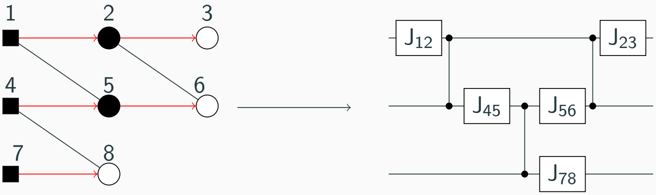
Extraction: causal flow



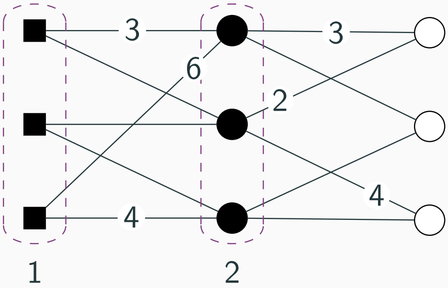
Extraction: causal flow



Extraction: causal flow

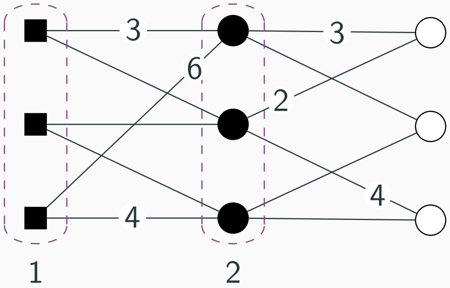


CV-flow to causal flow



$$A_{1\to 2} = \begin{pmatrix} 3 & 0 & 6 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix}$$

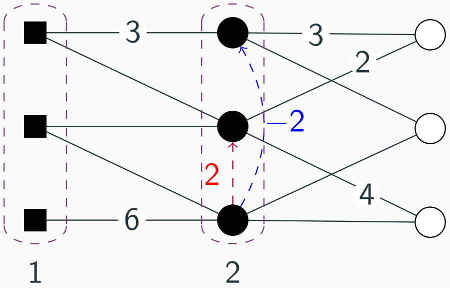
CV-flow to causal flow



$$C_3 \leftarrow C_3 {\color{blue}-2C_1} + {\color{red}2C_2}$$

$$A_{1\to 2} \sim \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 6 \end{pmatrix}$$

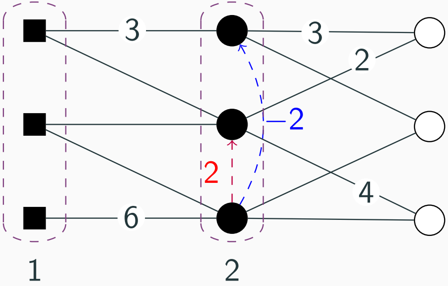
CV-flow to causal flow



$$C_3 \leftarrow C_3 {\color{blue}-2C_1} + {\color{red}2C_2}$$

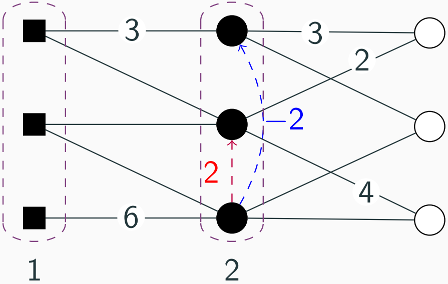
$$A_{1\to 2} \sim \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 6 \end{pmatrix}$$

CV-flow to causal flow



$$A_{2 \to O} = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

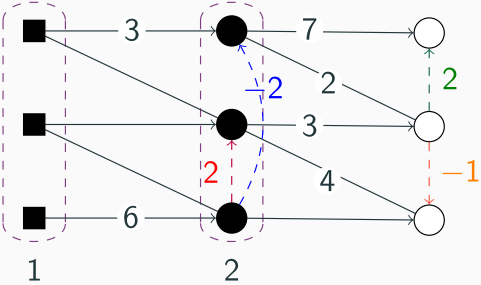
CV-flow to causal flow



$$C_2 \leftarrow C_2 - C_3 + 2\,C_1$$

$$A_1 \sim \begin{pmatrix} 3 & 2 & 0 \\ 7 & 0 & 0 \\ 0 & 4 & 6 \end{pmatrix} \sim \begin{pmatrix} 7 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 4 & 6 \end{pmatrix}$$

CV-flow to causal flow



$$C_2 \leftarrow C_2 - C_3 + 2C_1$$

$$A_1 \sim \begin{pmatrix} 3 & 2 & 0 \\ 7 & 0 & 0 \\ 0 & 4 & 6 \end{pmatrix} \sim \begin{pmatrix} 7 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 4 & 6 \end{pmatrix}$$

Extraction: CV-flow

Extraction: CV-flow

- **Lemma**: if a graph has CV-flow, then it can be split into layers, and each of these layers has a causal flow up to CX gates

Extraction: CV-flow

- **Lemma**: if a graph has CV-flow, then it can be split into layers, and each of these layers has a causal flow up to CX gates

- **Lemma**: these CX gates don't get in the way of the MBQC procedure

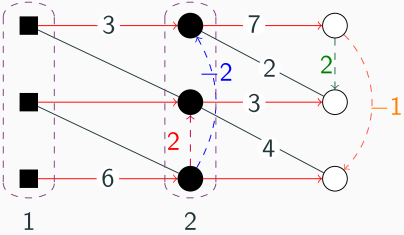
Extraction: CV-flow

- **Lemma**: if a graph has CV-flow, then it can be split into layers, and each of these layers has a causal flow up to CX gates
- **Lemma**: these CX gates don't get in the way of the MBQC procedure
- **Lemma**: these causal flows each have path covers, and the endpoints of paths in subsequent layers match up

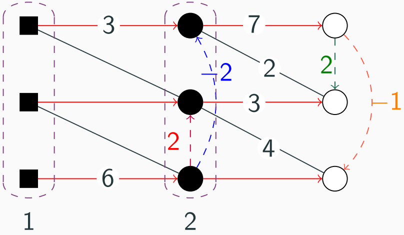
Extraction: CV-flow

- **Lemma**: if a graph has CV-flow, then it can be split into layers, and each of these layers has a causal flow up to CX gates
- **Lemma**: these CX gates don't get in the way of the MBQC procedure
- **Lemma**: these causal flows each have path covers, and the endpoints of paths in subsequent layers match up
- **Conclusion**: the CV-flow procedure can be broken down into a sequence of causal flow circuits with intermediate CX gates

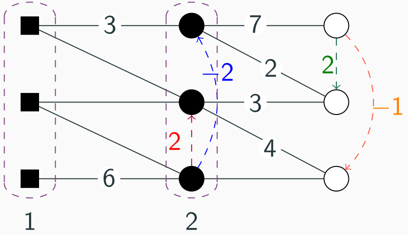
CV-flow: extraction

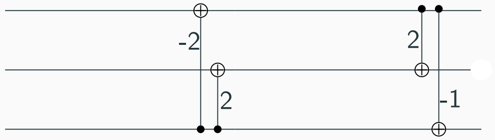


CV-flow: extraction

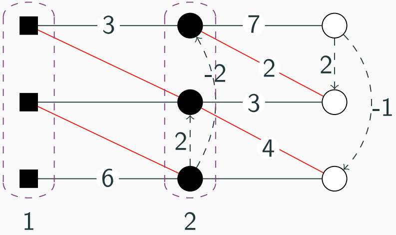


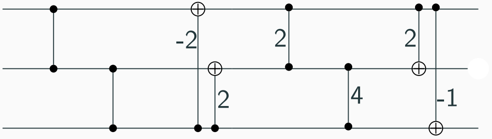
CV-flow: extraction



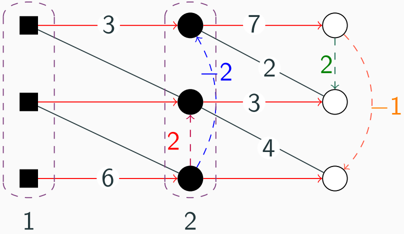


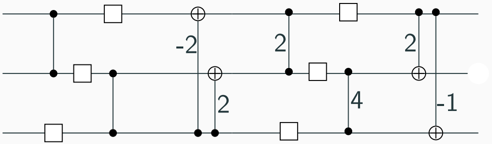
CV-flow: extraction



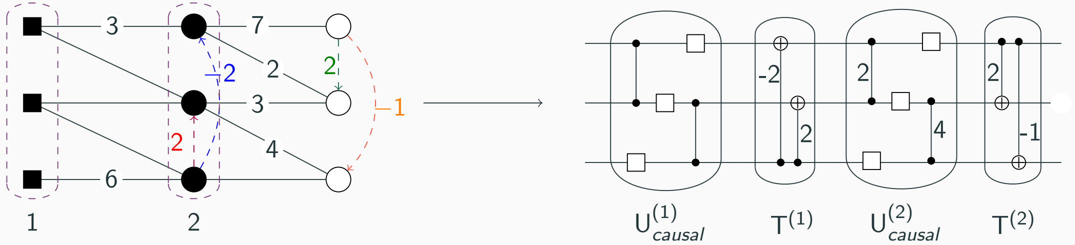


CV-flow: extraction





CV-flow: extraction



Convergence result

Formally, for any choice of input $\rho \in \mathcal{H}^{\otimes |O|}$ and any *local* unitaries from the gate set,

$$\lim_{\substack{\eta \to \infty \\ \delta \to 0}} \mathscr{M}_\eta^\delta[\rho] = \prod_{i \in \text{Layers}} \mathsf{T}^{(i)} \mathsf{U}^{(i)}_{\text{causal}} \left[\rho \right]$$

Takeaway: results

- MBQC procedure based on CV-flow, which converges iff $|I| = |O|$.
- Explicit circuit extraction scheme for the MBQC protocol when the graph state has CV-flow.
- (Poly-time algorithm for finding a CV-flow if one exists.)
- (Examples graphs that separate CV-flow from gflow)

Mehdi Mhalla and Simon Perdrix. "Finding Optimal Flows Efficiently". In: *Automata, Languages and Programming*. Ed. by Luca Aceto et al. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer, 2008, pp. 857–868.

Some open questions

1. A (pseudo-)necessity theorem?
2. Several measurement planes?
3. Infinite fields?
4. Rings?
5. Hypergraphs?

Thanks for watching!