F-flow: determinism in measurement-based quantum computing for qudits

Robert Booth, Aleks Kissinger, Damian Markham, Simon Perdrix

April 16th, 2021

Causal flow and gflow

- Vincent Danos and Elham Kashefi. "Determinism in the One-Way Model". In: *Physical Review A* 74.5 (Nov. 2006), p. 052310
- D. E. Browne et al. "Generalized Flow and Determinism in Measurement-Based Quantum Computation". en. In: *New Journal of Physics* 9.8 (Aug. 2007), pp. 250–250

Some motivation

- Anne Broadbent and Elham Kashefi. “Parallelizing Quantum Circuits”. en. In: *Theoretical Computer Science* 410.26 (June 2009), pp. 2489–2510
- Ross Duncan et al. “Graph-Theoretic Simplification of Quantum Circuits with the ZX-Calculus”. en. In: *Quantum* 4 (June 2020), p. 279
- Atul Mantri et al. “Flow Ambiguity: A Path Towards Classically Driven Blind Quantum Computation”. en. In: *Physical Review X* 7.3 (July 2017)

Quantum computing with qudits: states

- State space $\rightarrow$ d-dimensional Hilbert space:

$$\mathcal{H} = \mathrm{L}^2(\mathbb{Z}_d) \cong \mathbb{C}^d$$

$$|\psi\rangle \in \mathcal{H} \iff |\psi\rangle = \sum_{x \in \mathbb{Z}_d} c_x \, |x\rangle$$

- d is prime $\implies$ $\mathbb{Z}_d$ is a field, hence the name $\mathbb{F}$-flow

Quantum computing with qudits: gates

- Pauli gates

$$Z|x\rangle = e^{\frac{i2\pi x}{d}}|x\rangle = \omega^x|x\rangle \quad X|x\rangle = |x+1\rangle$$

$$\text{generic Pauli} \quad \rightsquigarrow \quad X^a Z^b \quad \text{for} \quad a, b \in \mathbb{Z}_d$$

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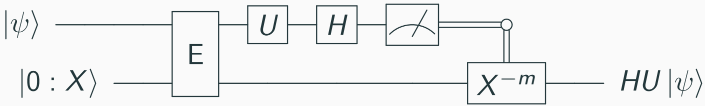
- Controlled phase or CZ-gate

$$E^w|x\rangle \otimes |y\rangle = \omega^{wxy}|x\rangle \otimes |y\rangle \quad \text{for} \quad w \in \mathbb{Z}_d$$

Gate teleportation protocols



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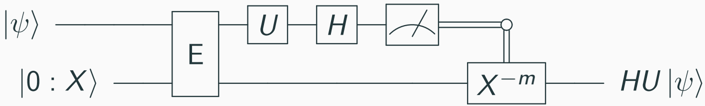

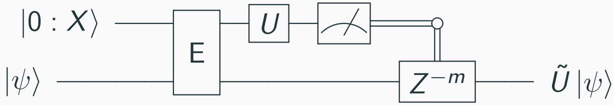

Quantum computing with qudits: measurements

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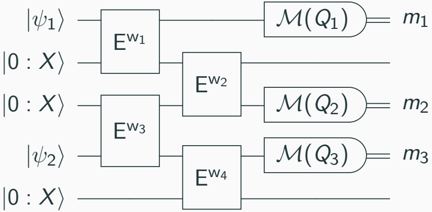
$$\mathcal{M}(Q) := \{M \text{ fixpoint unitary} \mid QM = \omega MQ\}$$

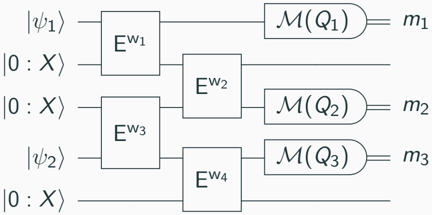
- If $M \in \mathcal{M}(Q)$, then M has d eigenvalues, and $Q\,|k : M\rangle = |k+1 : M\rangle$.

MBQC protocols



- Problem 1: runnability?
- Problem 2: determinism?

Open graph states



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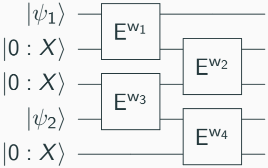

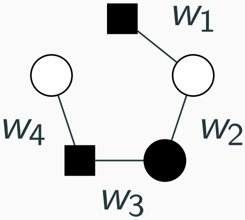

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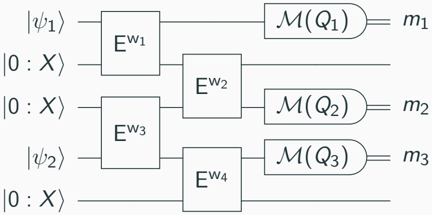
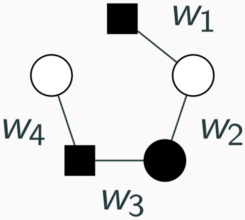
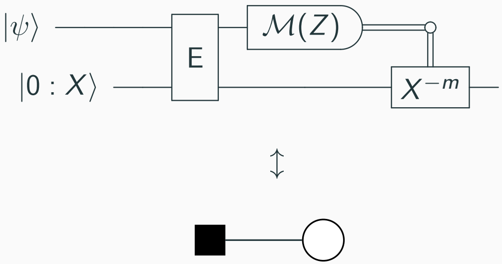




Graph states (cont.)



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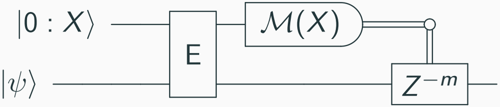
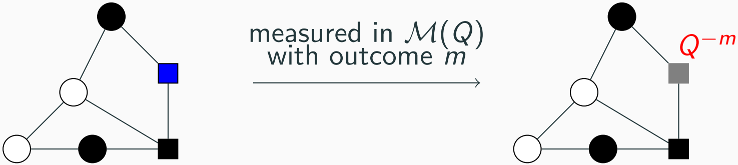


$$\updownarrow$$

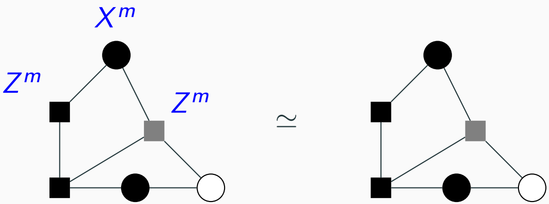
Measurement errors



$$\langle m : M| = \langle 0 : M|\, Q^{-m}$$

Corrections

1.



$$X^m \prod_{neighbours} Z^m \lvert G\rangle = \lvert G\rangle$$

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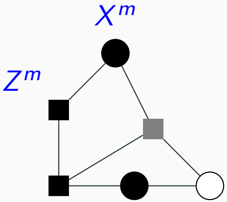
1. 

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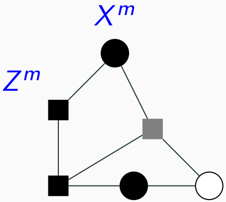

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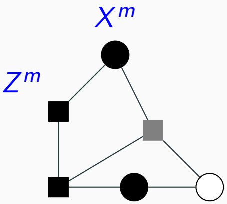
1. 

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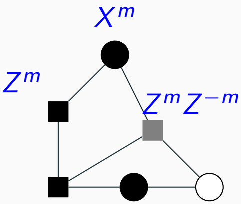

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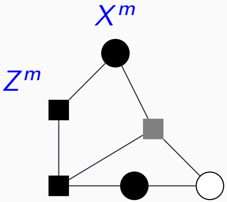

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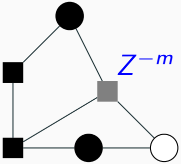

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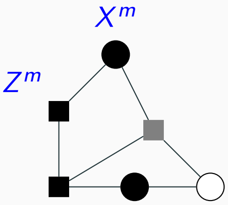
1. 

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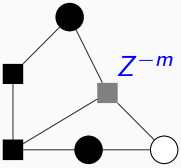
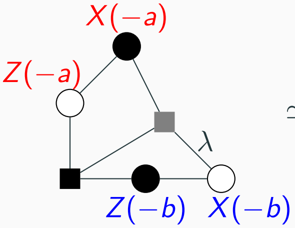


2. 

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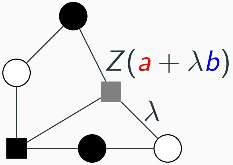
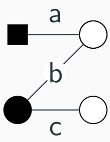


$\mathbb{F}$-flow (simplified)

The open graph G has an $\mathbb{F}$-flow if there is a lower triangular (square) matrix C such that $A_G C$ is lower triangular.

F-flow: an example



$$\underbrace{\begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & b & c \\ a & b & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix}}_{A_G}$$

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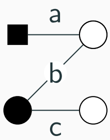

$$\underbrace{\begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & b & c \\ a & b & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix}}_{A_G} \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{C}$$

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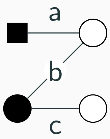

$$\underbrace{\begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & b & c \\ a & b & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix}}_{A_G} \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ ba^{-1} & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix}$$

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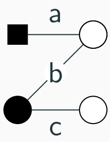

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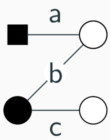

$$\underbrace{\begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & b & c \\ a & b & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix}}_{A_G} \underbrace{\begin{pmatrix} \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} \\ 0 & \color{red}{1} & \color{red}{0} & \color{red}{0} \\ a^{-1} & 0 & \color{red}{0} & \color{red}{0} \\ 0 & 0 & 0 & \color{red}{0} \end{pmatrix}}_{C} = \begin{pmatrix} \color{red}{1} & \color{red}{0} & \color{red}{0} & \color{red}{0} \\ ba^{-1} & \color{red}{0} & \color{red}{0} & \color{red}{0} \\ 0 & b & \color{red}{0} & \color{red}{0} \\ 0 & c & 0 & \color{red}{0} \end{pmatrix}$$

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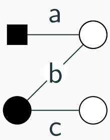

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Main results

Theorem

Suppose an open graph state has $\mathbb{F}$-flow, then the corresponding MBQC is runnable and robustly deterministic. Furthermore, for a given choice of measurements, it realises a partial isometry $\mathcal{H}^{\otimes I} \to \mathcal{H}^{\otimes O}$.

Theorem

If $\mathbb{F} = \mathbb{Z}_3$, and the MBQC is runnable and robustly deterministic, then the underlying open graph has a $\mathbb{Z}_3$-flow.

Auxiliary results

- A poly-time algorithm for finding an optimal-depth $\mathbb{F}$-flow for an open graph whenever it has one.
- Some results on manipulating open graphs with $\mathbb{F}$-flows: connecting $\mathbb{F}$-flows, deleting vertices, local-Clifford operations.
- (Ancilla-less extraction algorithm when all the measurements are $\mathcal{M}(Z)$).

Future work

1. Ancilla-less circuit extraction algorithm for all measurement spaces.
2. Pauli $\mathbb{F}$-flow?
3. Other rings?

Thanks for watching!

F-flow (full)

Definition

(G, I, O, λ) has an if there exists $C \in \mathbb{F}^{V \times V}$ and a permutation matrix $Q \in \mathbb{Z}_2^{V \times V}$ such that

1. $\forall u \in O^c$, $\lambda(u) = (C_{uu}, (QGQ^\mathsf{T} C)_{uu})$;
2. $C_{uv} = 0$ whenever $u \in I$ or $v \in O$;
3. C and $QGQ^\mathsf{T} C$ are lower triangular.