Outcome determinism in measurement-based quantum computing with qudits

Robert Booth, Aleks Kissinger, Damian Markham, Clément Meignant, Simon Perdrix

QuPa, June 17th 2021

Causal flow and gflow

- Vincent Danos and Elham Kashefi. "Determinism in the One-Way Model". In: *Physical Review A* 74.5 (Nov. 2006), p. 052310
- D. E. Browne et al. "Generalized Flow and Determinism in Measurement-Based Quantum Computation". en. In: *New Journal of Physics* 9.8 (Aug. 2007), pp. 250–250

Some motivation

- Anne Broadbent and Elham Kashefi. “Parallelizing Quantum Circuits”. en. In: *Theoretical Computer Science* 410.26 (June 2009), pp. 2489–2510
- Ross Duncan et al. “Graph-Theoretic Simplification of Quantum Circuits with the ZX-Calculus”. en. In: *Quantum* 4 (June 2020), p. 279
- Atul Mantri et al. “Flow Ambiguity: A Path Towards Classically Driven Blind Quantum Computation”. en. In: *Physical Review X* 7.3 (July 2017)
- Robert I. Booth and Damian Markham. “Flow Conditions for Continuous Variable Measurement-Based Quantum Computing”. In: *arXiv:2104.00572 [quant-ph]* (Apr. 2021)

Quantum computing with qudits: states

- State space $\to$ d-dimensional Hilbert space:

$$\mathcal{H} = \mathrm{L}^2(\mathbb{Z}_d) \cong \mathbb{C}^d$$

$$|\psi\rangle \in \mathcal{H} \iff |\psi\rangle = \sum_{x \in \mathbb{Z}_d} c_x |x\rangle$$

- d is prime $\implies$ $\mathbb{Z}_d$ is a field

Quantum computing with qudits: gates

- Pauli gates

$$Z|x\rangle = e^{\frac{i2\pi x}{d}}|x\rangle = \omega^x|x\rangle \quad X|x\rangle = |x+1\rangle$$

$$\text{generic Pauli} \quad \rightsquigarrow \quad X^a Z^b \quad \text{for} \quad a, b \in \mathbb{Z}_d$$

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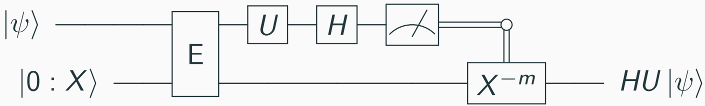
- These gates are not self-inverse

$$(X^a Z^b)^d = 1_{\mathcal{H}} \qquad H^4 = 1_{\mathcal{H}} \qquad E^d = 1_{\mathcal{H}\otimes\mathcal{H}}.$$

Gate teleportation protocols



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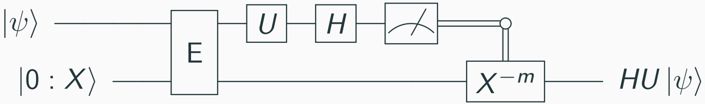

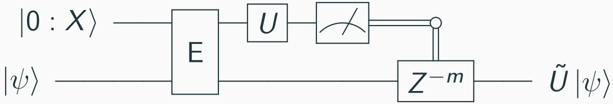

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Lemma

$M, N \in \mathcal{M}(Q)$ if and only if there is $U \in SU(d)$ such that $M = UNU^\dagger$ and furthermore $[U, Q] = 0$.

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- $X^{m_v}_u$ and $Z^{m_v}_u$: Pauli corrections depending on the outcome m_v of the measurement of vertex v.

Open graphs

$$\mathfrak{P} \simeq \left(\prod_{v \in O^c}^{\prec} X_{x(v)}^{m_v} Z_{z(v)}^{m_v} M_v^{a_v, b_v}(\vec{\theta}_v) \right) \left(\prod_{(u,v) \in G} E_{u,v}^{G_{uv}} \right) \left(\prod_{v \in I^c} N_v \right)$$

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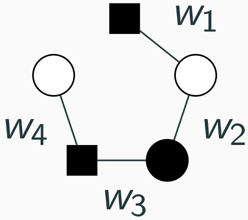

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Outcome determinism

$$\mathfrak{P} : \rho \longmapsto \sum_{\vec{m} \in \mathbb{Z}_d^{O^c}} A_{\vec{m}}(\theta) \rho A_{\vec{m}}(\theta)^* \tag{1}$$

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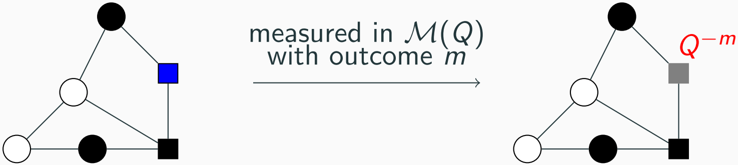
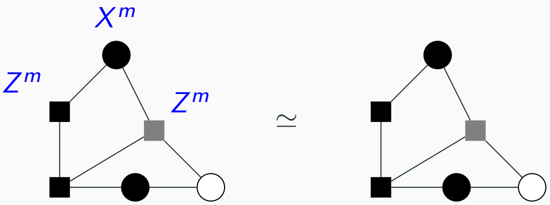
- It is robustly deterministic if it is uniformly and stepwise strongly deterministic.

Measurement errors



$$\langle m : M| = \langle 0 : M|\, Q^{-m}$$

Corrections

1. 

$$X^m \prod_{\textit{neighbours}} Z^m |G\rangle = |G\rangle$$

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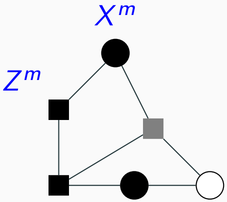

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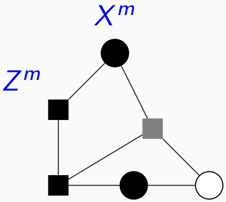

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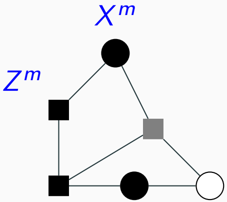

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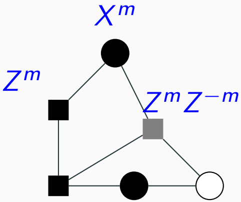

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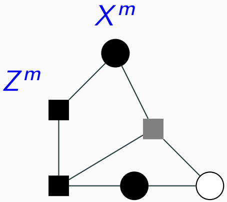
1. 

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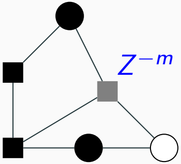

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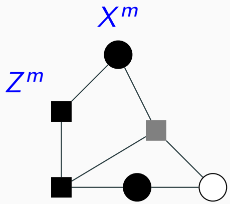

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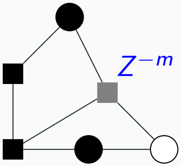

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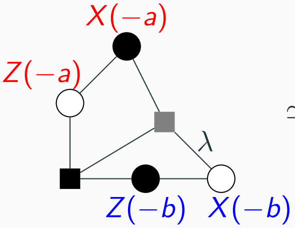

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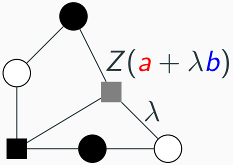
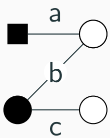


F-flow (simplified)

The open graph G has an $\mathbb{F}$-flow if there is a lower triangular (square) matrix C such that $A_G C$ is lower triangular.



$$\underbrace{\begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & b & c \\ a & b & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix}}_{A_G}$$

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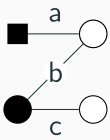

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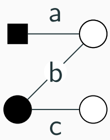

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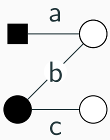

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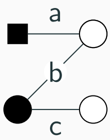

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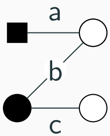

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Main results

Theorem

Suppose an open graph has $\mathbb{Z}_d$-flow, then the corresponding MBQC is runnable and robustly deterministic. Furthermore, for a given choice of measurements, it realises a partial isometry $\mathcal{H}^{\otimes I} \to \mathcal{H}^{\otimes O}$.

Theorem

If an MBQC is runnable and robustly deterministic, then the underlying open graph has a $\mathbb{Z}_d$-flow.

Auxiliary results

- A poly-time algorithm for finding an optimal-depth $\mathbb{Z}_d$-flow for an open graph whenever it has one.
- Some results on manipulating open graphs with $\mathbb{Z}_d$-flows: connecting $\mathbb{Z}_d$-flows, deleting vertices, local-Clifford operations.
- (Ancilla-less quantum circuit extraction algorithm when all the measurements are $\mathcal{M}(Z)$).

Future work

1. Ancilla-less circuit extraction algorithm for all measurement spaces.
2. Pauli $\mathbb{F}$-flow?
3. Other groups/rings? In particular: $A \cong \prod \mathbb{Z}_{p^k}^n$
   $\mathbb{F}$-flow takes care of factors $\mathbb{Z}_p^n$

Thanks for watching!

F-flow (full)

Definition

(G, I, O, λ) has an $\mathbb{F}$-flow if there exists $C \in \mathbb{F}^{V \times V}$ and a permutation matrix $Q \in \mathbb{Z}_2^{V \times V}$ such that

1. $\forall u \in O^c$, $\lambda(u) = (C_{uu}, (QGQ^\mathsf{T} C)_{uu})$;
2. $C_{uv} = 0$ whenever $u \in I$ or $v \in O$;
3. C and $QGQ^\mathsf{T} C$ are lower triangular.

A technical lemma

Lemma

Let $|\phi\rangle, |\phi'\rangle$ *be two states of a register* V *of qudits,* Q *a Pauli operator and fix some* $v \in V$. *If for every measurement* $M \in \mathcal{M}(Q_u)$ *of the qudit* u *and every* $m \in \mathbb{Z}_d$*, we have*

$$\langle m : M \mid \phi\rangle \simeq \langle m : M \mid \phi'\rangle \quad \text{and} \quad \|\langle m : M \mid \phi\rangle\| = \frac{1}{\sqrt{d}} = \|\langle m : M \mid \phi'\rangle\|, \tag{2}$$

then at least one of the following holds:

1. $|\phi\rangle \simeq |\phi'\rangle$;
2. $|\phi\rangle$ *and* $|\phi'\rangle$ *are separable and there are* $x, y \in \mathbb{Z}_d$ *and* $|\psi\rangle \in \mathcal{H}_{V\setminus\{v\}}$ *such that*

$$|\phi\rangle = |x : Q\rangle_v \otimes |\psi\rangle \quad \text{and} \quad \left|\phi'\right\rangle \simeq |y : Q\rangle_v \otimes |\psi\rangle, \tag{3}$$

where $|x : Q\rangle$ *is the eigenvector of* Q *associated with eigenvalue* ω^x.