# Outcome determinism in measurement-based quantum computing with qudits

*Robert Booth*, Aleks Kissinger, Damian Markham, Clément Meignant, Simon Perdrix

ZX meeting, June 21st 2021

# Causal flow and gflow

- Vincent Danos and Elham Kashefi. "Determinism in the One-Way Model". In: *Physical Review A* 74.5 (Nov. 2006), p. 052310
- D. E. Browne et al. "Generalized Flow and Determinism in Measurement-Based Quantum Computation". en. In: *New Journal of Physics* 9.8 (Aug. 2007), pp. 250–250

# Some motivation


- Anne Broadbent and Elham Kashefi. "Parallelizing Quantum Circuits". en. In: *Theoretical Computer Science* 410.26 (June 2009), pp. 2489–2510
- Ross Duncan et al. "Graph-Theoretic Simplification of Quantum Circuits with the ZX-Calculus". en. In: *Quantum* 4 (June 2020), p. 279
- Atul Mantri et al. "Flow Ambiguity: A Path Towards Classically Driven Blind Quantum Computation". en. In: *Physical Review X* 7.3 (July 2017)
- Robert I. Booth and Damian Markham. "Flow Conditions for Continuous Variable Measurement-Based Quantum Computing". In: *arXiv:2104.00572 [quant-ph]* (Apr. 2021)

# Quantum computing with qudits: states

- State space $\to$ $d$-dimensional Hilbert space:

$$\mathcal{H} = \mathrm{L}^2(\mathbb{Z}_d) \cong \mathbb{C}^d$$

$$|\psi\rangle \in \mathcal{H} \iff |\psi\rangle = \sum_{x \in \mathbb{Z}_d} c_x |x\rangle$$

- $d$ is prime $\implies$ $\mathbb{Z}_d$ is a field

# Quantum computing with qudits: gates

- Pauli gates

$$Z|x\rangle = e^{\frac{i2\pi x}{d}}|x\rangle = \omega^x|x\rangle \quad X|x\rangle = |x+1\rangle$$

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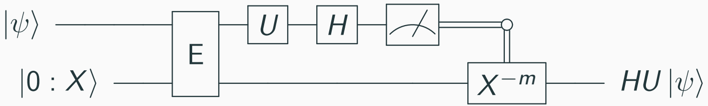
- These gates are not self-inverse

$$(X^a Z^b)^d = 1_{\mathcal{H}} \qquad H^4 = 1_{\mathcal{H}} \qquad E^d = 1_{\mathcal{H}\otimes\mathcal{H}}.$$

# Gate teleportation protocols



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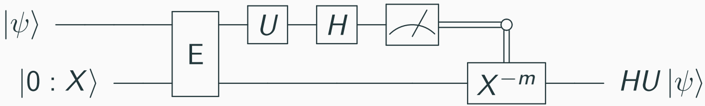


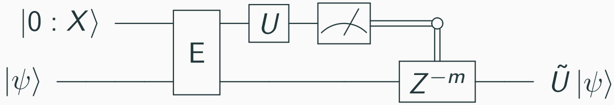

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**Lemma**

*$M, N \in \mathcal{M}(Q)$ if and only if there is $U \in SU(d)$ such that $M = UNU^\dagger$ and furthermore $[U, Q] = 0$.*

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- $X^{m_v}_u$ and $Z^{m_v}_u$ : Pauli corrections depending on the outcome $m_v$ of the measurement of vertex $v$.

# Open graphs

$$\mathfrak{P} \simeq \left( \prod_{v \in O^c}^{\prec} X_{\mathbf{x}(v)}^{m_v} Z_{\mathbf{z}(v)}^{m_v} M_v^{a_v, b_v}(\vec{\theta}_v) \right) \left( \prod_{(u,v) \in G} E_{u,v}^{G_{uv}} \right) \left( \prod_{v \in I^c} N_v \right)$$

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Vincent Danos, Elham Kashefi, and Prakash Panangaden. "The Measurement Calculus". In: *Journal of the ACM* 54.2 (Apr. 2007), 8–es.

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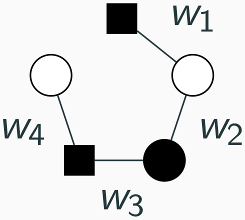


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# Outcome determinism

$$\mathfrak{P} : \rho \longmapsto \sum_{\vec{m} \in \mathbb{Z}_d^{O^c}} A_{\vec{m}}(\theta) \rho A_{\vec{m}}(\theta)^*$$

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- It is *strongly deterministic* if all the maps in it's Kraus decomposition are equal up to a phase.
- It is robustly deterministic if it is uniformly and stepwise strongly deterministic.

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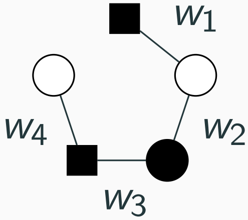


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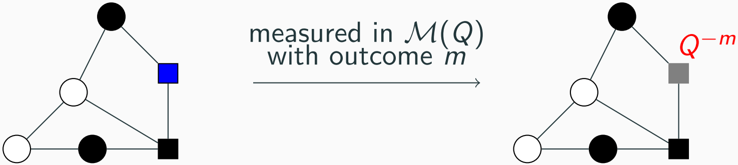
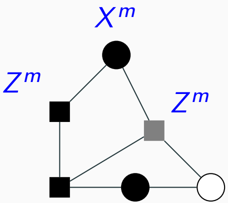
# Flow conditions, graphically

# Measurement errors




$$\langle m : M| = \langle 0 : M|\, Q^{-m}$$

# Corrections

1. 

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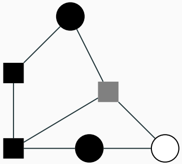

$$X^m \prod_{neighbours} Z^m \left|G\right\rangle = \left|G\right\rangle$$

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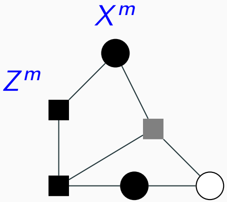
1. 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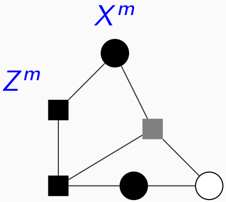

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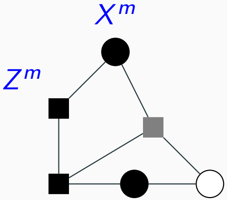


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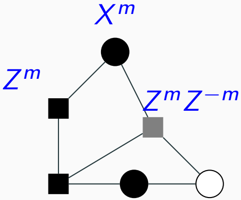

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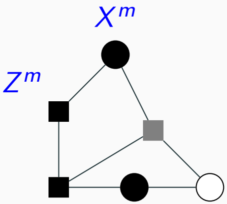
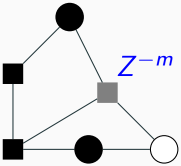
1.  $\simeq$ 

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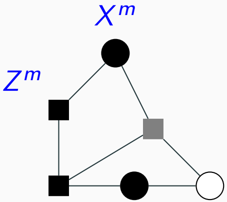


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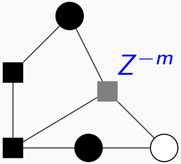


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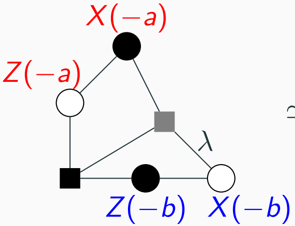


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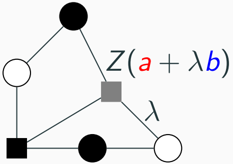
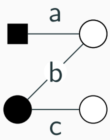


# F-flow (simplified)

The open graph $G$ has an $\mathbb{F}$-flow if there is a lower triangular (square) matrix $C$ such that $A_G C$ is lower triangular.

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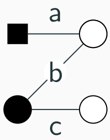


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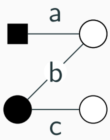


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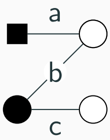


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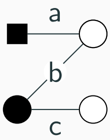


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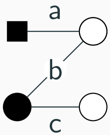


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# F-flow (full)

**Definition**

$(G, I, O, \lambda)$ has an $\mathbb{F}$-flow if there exists $C \in \mathbb{F}^{V \times V}$ and a permutation matrix $Q \in \mathbb{Z}_2^{V \times V}$ such that

1. $\forall u \in O^c$, $\lambda(u) = (C_{uu}, (QGQ^\top C)_{uu})$;
2. $C_{uv} = 0$ whenever $u \in I$ or $v \in O$;
3. $C$ and $QGQ^\top C$ are lower triangular.

# Main results: necessary and sufficient

**Theorem**

*Suppose an open graph has $\mathbb{Z}_d$-flow, then the corresponding MBQC is runnable and robustly deterministic. Furthermore, for a given choice of measurements, it realises an isometry $\mathcal{H}^{\otimes I} \to \mathcal{H}^{\otimes O}$.*

**Theorem**

*If an MBQC is runnable and robustly deterministic, then the underlying open graph has a $\mathbb{Z}_d$-flow.*

# Proof of converse

$$|G(\phi)\rangle \longmapsto X^m_{\mathbf{x}(u)} Z^m_{\mathbf{z}(u)} \left( \langle m : \mathbf{M}(u)| \bigotimes_{v \in O^c \setminus \{u\}} \langle 0 : \mathbf{M}(v)| \right) |G(\phi)\rangle$$

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&= \left( \bigotimes_{v \in O^c} \langle 0 : \mathbf{M}(v)| \right) X^m_{\mathbf{x}(u)} Z^m_{\mathbf{z}(u)} Q^m_u |G(\phi)\rangle \\
&= \bigotimes_{v \in O^c} \langle 0 : \mathbf{M}(v)|G(\phi)\rangle .
\end{aligned}
$$

# A technical lemma

**Lemma**

*Let $|\phi\rangle, |\phi'\rangle$ be two states of a register $V$ of qudits, $Q$ a Pauli operator and fix some $v \in V$. If for every measurement $M \in \mathcal{M}(Q_u)$ of the qudit $u$ and every $m \in \mathbb{Z}_d$, we have*

$$\langle m : M \mid \phi\rangle \simeq \langle m : M \mid \phi'\rangle \quad \text{and} \quad \|\langle m : M \mid \phi\rangle\| = \frac{1}{\sqrt{d}} = \|\langle m : M \mid \phi'\rangle\|, \tag{1}$$

*then at least one of the following holds:*

1. $|\phi\rangle \simeq |\phi'\rangle$;
2. $|\phi\rangle$ *and* $|\phi'\rangle$ *are separable and there are* $x, y \in \mathbb{Z}_d$ *and* $|\psi\rangle \in \mathcal{H}_{V\setminus\{v\}}$ *such that*

$$|\phi\rangle = |x : Q\rangle_v \otimes |\psi\rangle \quad \text{and} \quad \left|\phi'\right\rangle \simeq |y : Q\rangle_v \otimes |\psi\rangle\,, \tag{2}$$

*where $|x : Q\rangle$ is the eigenvector of $Q$ associated with eigenvalue $\omega^x$.*

# Proof of converse (cont.)

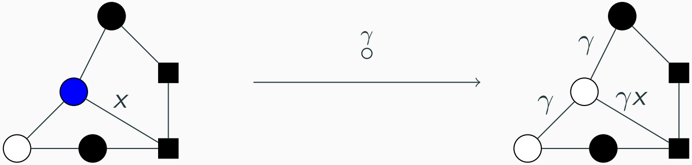
$$\left(\bigotimes_{v \in O^c} \langle 0 : \mathbf{M}(v)|\right) X^m_{\mathbf{x}(u)} Z^m_{\mathbf{z}(u)} Q^m_u \,|G(\phi)\rangle = \bigotimes_{v \in O^c} \langle 0 : \mathbf{M}(v)|G(\phi)\rangle \,.$$

$$\Downarrow$$

$$X^m_{\mathbf{x}(u)} Z^m_{\mathbf{z}(u)} Q^m_u \,|G(\phi)\rangle = |G(\phi)\rangle \,.$$

# Manipulating flows

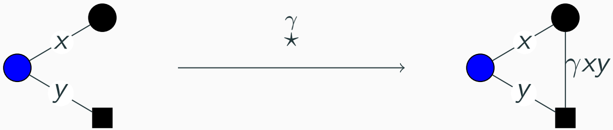
# Local scaling




$$G \overset{\gamma}{\circ} w = D_\gamma^w G D_\gamma^w \quad \text{and} \quad C^{\overset{\gamma}{\circ} w} = D_{\gamma^{-1}}^w C$$

# Local complementation

$$G \overset{\gamma}{\star} w := G + \gamma G 1_{\{w\}} 1_{\{w\}}^{\mathsf{T}} G - \gamma D^2_{G e_w}.$$



# Other transformations

- pasting two $\mathbb{F}$-flows together;
- removing vertices;
- focusing.

## Auxiliary results

- A poly-time algorithm for finding an optimal-depth $\mathbb{Z}_d$-flow for an open graph whenever it has one.
- (Ancilla-less quantum circuit extraction algorithm when all the measurements are $\mathcal{M}(Z)$).

# Future work

1. Ancilla-less circuit extraction algorithm for all measurement spaces.
2. Pauli $\mathbb{F}$-flow?
3. Other groups/rings? In particular: $A \cong \prod \mathbb{Z}_{p^k}^n$
   $\mathbb{F}$-flow takes care of factors $\mathbb{Z}_p$

Thanks for watching!