Complete ZX-calculi for the stabiliser fragment in odd prime dimensions

Robert I. Booth, Titouan Carette

ZX meeting, 9 June 2022

Previous work

- Miriam Backens. "The ZX-calculus Is Complete for Stabilizer Quantum Mechanics". In: *New Journal of Physics* 16.9 (Sept. 17, 2014), p. 093021
- Miriam Backens, Simon Perdrix, and Quanlong Wang. "A Simplified Stabilizer ZX-calculus". In: *Electronic Proceedings in Theoretical Computer Science* 236 (Jan. 1, 2017), pp. 1–20
- Quanlong Wang and Xiaoning Bian. "Qutrit Dichromatic Calculus and Its Universality". In: *Electronic Proceedings in Theoretical Computer Science* 172 (Dec. 28, 2014), pp. 92–101
- Quanlong Wang. "Qutrit ZX-calculus Is Complete for Stabilizer Quantum Mechanics". In: *Electronic Proceedings in Theoretical Computer Science* 266 (Feb. 27, 2018), pp. 58–70

Quantum computing with qupits: states

- State space $\rightarrow$ p-dimensional Hilbert space:

$$\mathcal{H} = \mathrm{L}^2(\mathbb{Z}_p) \cong \mathbb{C}^d$$

$$|\psi\rangle \in \mathcal{H} \iff |\psi\rangle = \sum_{x \in \mathbb{Z}_p} c_x |x\rangle$$

- p is prime $\implies$ $\mathbb{Z}_p$ is a field

Quantum computing with qupits: gates

- Pauli gates

$$Z\,|x\rangle = e^{\frac{i2\pi x}{p}}\,|x\rangle = \omega^x\,|x\rangle \qquad X\,|x\rangle = |x+1 \mod p\rangle$$

$$\text{generic Pauli} \quad \rightsquigarrow \quad X^a Z^b \quad \text{for} \quad a, b \in \mathbb{Z}_p$$

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- Hadamard gate

$$H|x\rangle = \sum_{k\in\mathbb{Z}_p}\omega^{kx}|x\rangle \quad \text{and} \quad HZH^\dagger = X$$

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- Controlled phase or CZ-gate

$$E|x\rangle \otimes |y\rangle = \omega^{xy}|x\rangle \otimes |y\rangle$$

Quantum computing with qupits: gates

- Pauli gates

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- Controlled phase or CZ-gate

$$E\left|x\right\rangle \otimes \left|y\right\rangle = \omega^{xy}\left|x\right\rangle \otimes \left|y\right\rangle$$

- Phase gate

$$S\left|x\right\rangle = \omega^{2^{-1}x(x+1)}\left|x\right\rangle$$

Quantum computing with qupits: gates (cont.)

These gates are not self-inverse

$$(X^a Z^b)^p = 1_{\mathcal{H}} \qquad S^p = 1_{\mathcal{H}} \qquad E^p = 1_{\mathcal{H}\otimes\mathcal{H}} \qquad H^4 = 1_{\mathcal{H}}.$$

The stabiliser fragment

- Recall the qubit stabiliser fragment:

  Pauli states + Clifford unitaries + Pauli projections

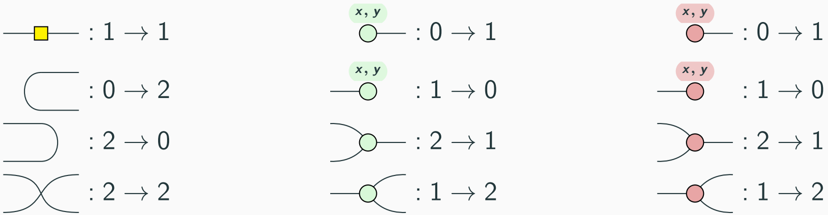
- The Clifford group on n qupits is the normaliser in $U(\mathcal{H}^{\otimes n})$ of the group of Paulis.

  Symplectic matrix + Translation + Global phase

- Formally, the Clifford group is a representation of $\mathrm{ASymp}(\mathbb{Z}_p^{2n}) \rtimes \mathbb{T}$

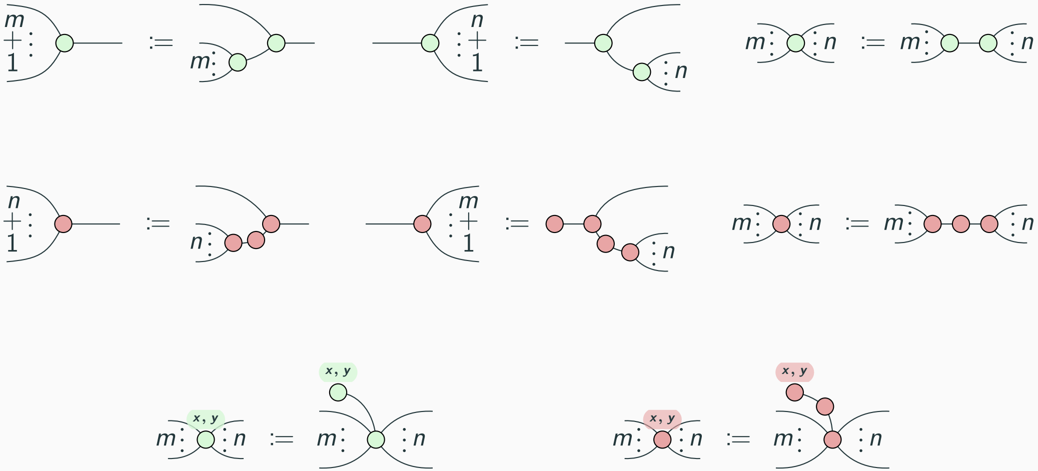
Generators of the calculus

Letting $x, y \in \mathbb{Z}_p$:

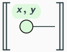


...and also their unloved sibling, $\star : 0 \to 0$.

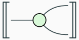
Spiders (and ants?)



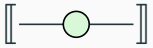
Standard interpretation



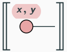
$$= \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \, |k : Z\rangle$$



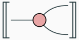
$$= \sum_{k\in\mathbb{Z}_p} |k : Z\rangle\langle k, k : Z|$$



$$= \sum_{k\in\mathbb{Z}_p} |k : Z\rangle\langle k : Z|$$



$$= \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \, |-k : X\rangle$$

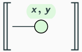


$$= \sum_{k\in\mathbb{Z}_p} |-k, -k : X\rangle\langle k : X|$$

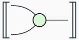
$$= \sum_{k\in\mathbb{Z}_p} |k : Z\rangle\langle k : Z|$$

$$= \sum_{k\in\mathbb{Z}_p} |kk : Z\rangle$$

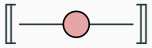
$$= \sum_{k,\ell\in\mathbb{Z}_p} |k, \ell : Z\rangle\langle \ell, k : Z|$$



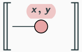
$$= \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \, \langle k : Z|$$



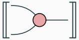
$$= \sum_{k\in\mathbb{Z}_p} |k, k : Z\rangle\langle k : Z|$$



$$= \sum_{k\in\mathbb{Z}_p} |-k : X\rangle\langle k : X|$$



$$= \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \, \langle k : X|$$



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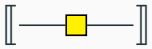
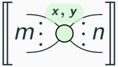


$$= \sum_{k\in\mathbb{Z}_p} |k : X\rangle\langle k : Z|$$

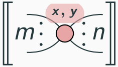
$$= \sum_{k\in\mathbb{Z}_p} \langle kk : Z|$$

$$[\![\star]\!] = -1$$

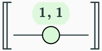
Standard interpretation



$$= \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \, |k:Z\rangle^{\otimes n} \langle k:Z|^{\otimes m}$$

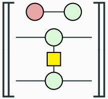


$$= \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \, |-k:X\rangle^{\otimes n} \langle k:X|^{\otimes m}$$



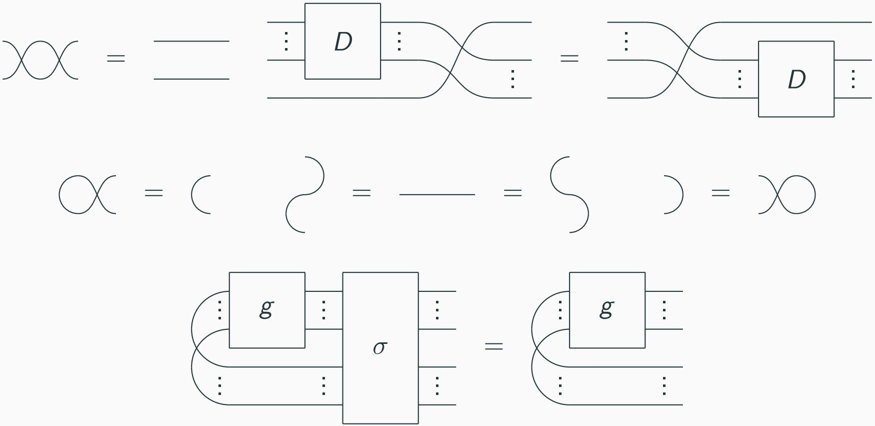
$= S$

$= H$



$= E$

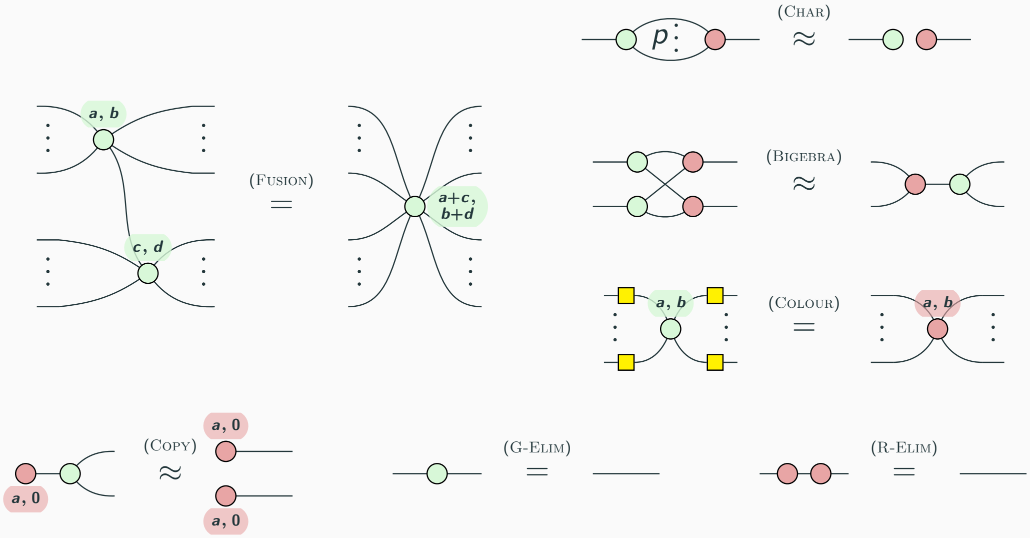
Structural axioms



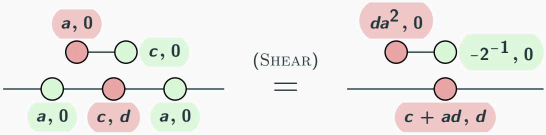
The calculus is flexsymmetric!

Titouan Carette. "Wielding the ZX-calculus, Flexsymmetry, Mixed States, and Scalable Notations". Theses. Université de Lorraine, Nov. 2021.

“Standard” ZX axioms

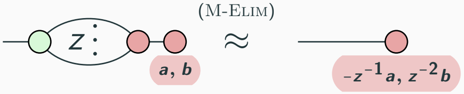


Axiom for affine shears

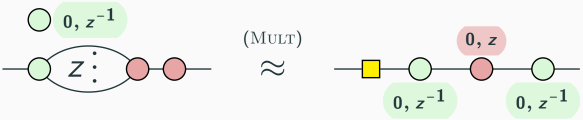


$$\begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ a \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} ad \\ a \end{bmatrix}$$

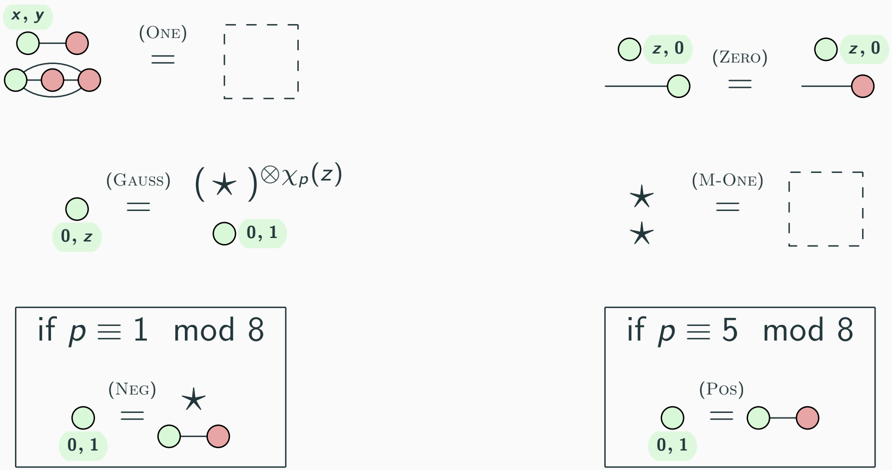
Axioms for multipliers



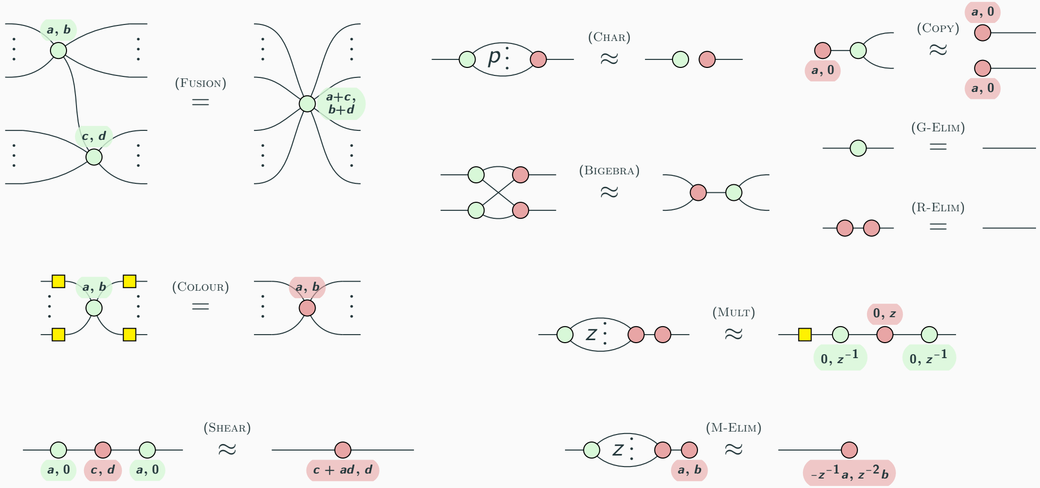
$$|k\rangle \mapsto |zk\rangle$$



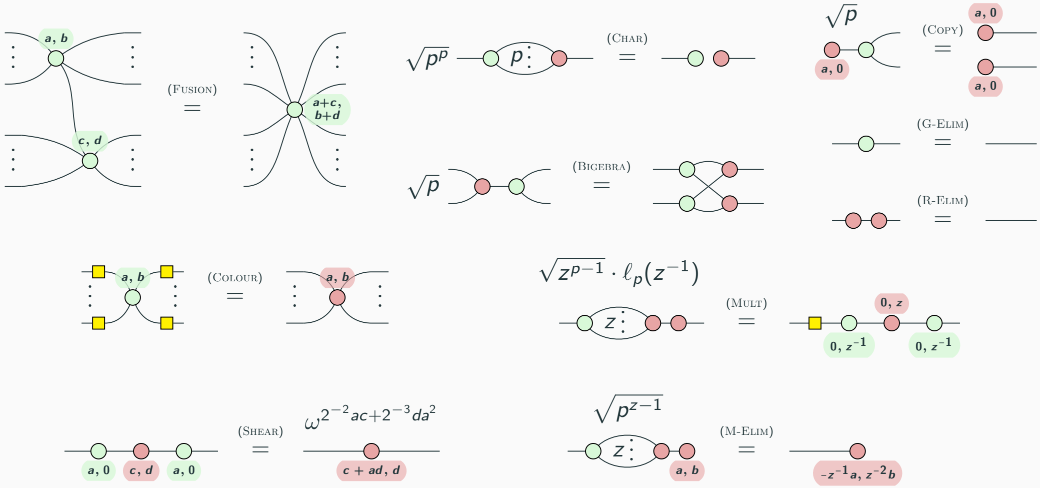
$$\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -z & -1 \end{bmatrix} \begin{bmatrix} 1 & z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$


x, y
(One)
z, 0
(Zero)
z, 0
(Gauss)
$(\star)^{\otimes \chi_p(z)}$
0, z
0, 1
(M-One)
if $p \equiv 1 \mod 8$
(Neg)
0, 1
if $p \equiv 5 \mod 8$
(Pos)
0, 1

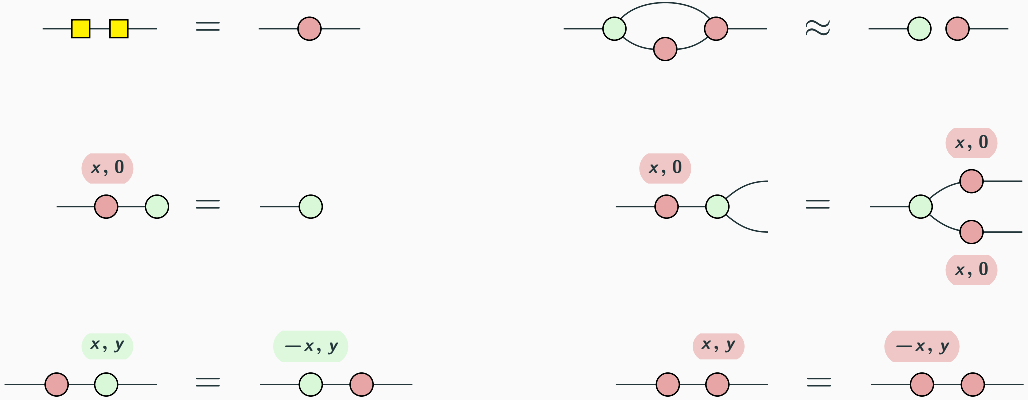
Scalarless axiomatisation



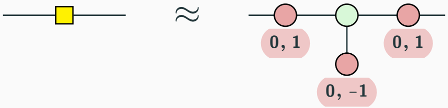
Scalarless axiomatisation (cont.)

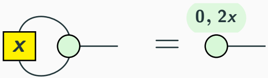


Elementary derivations

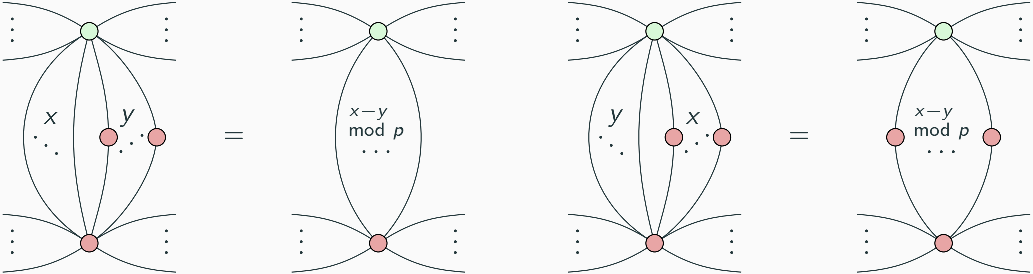


Elementary derivations (cont.)

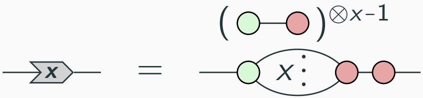




Spider wars



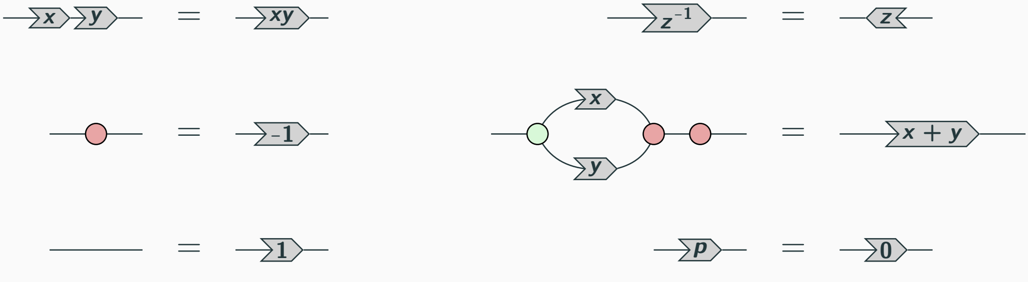
Syntactic sugar for multi-edges



Multipliers are not flexsymmetric!

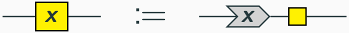
Fabio Zanasi. "Interacting Hopf Algebras: The Theory of Linear Systems". PhD thesis. Ecole Normale Superieure de Lyon, May 4, 2018.

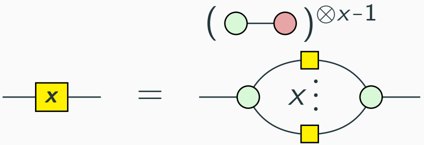
Syntactic sugar for multi-edges (cont.)



Multipliers are a presentation of the field $\mathbb{Z}_p$!

Weighted Hadamards

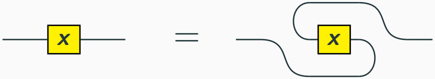


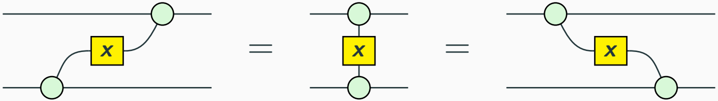


Alex Townsend-Teague and Konstantinos Meichanetzidis. "Classifying Complexity with the ZX-Calculus: Jones Polynomials and Potts Partition Functions". Mar. 11, 2021.

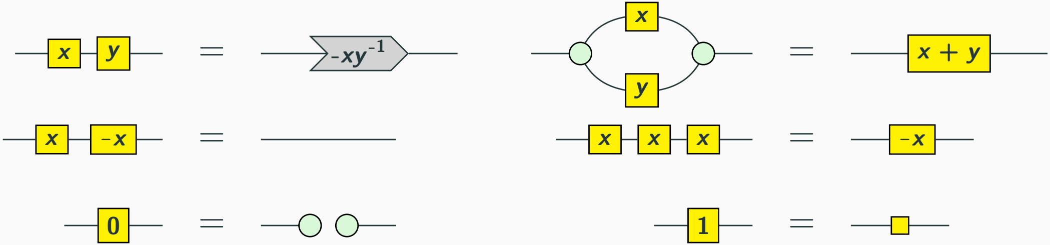
Weighted Hadamards (cont.)

Weighted Hadamards are flexsymmetric:

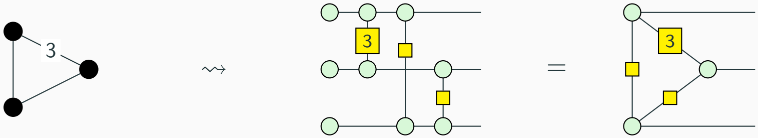




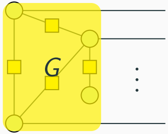
Weighted Hadamard equations



Graph states

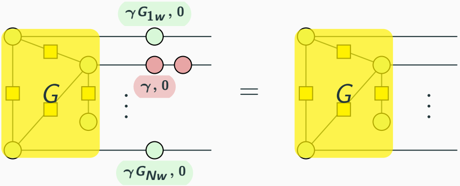


For an arbitrary graph G,

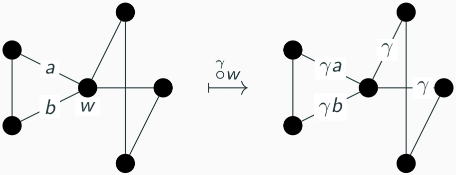


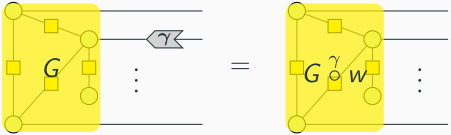
Pauli stabilisers

$$X_v^{\gamma} \prod_{w \in N(v)} Z_w^{\gamma G_{vw}} |G\rangle = |G\rangle \, .$$

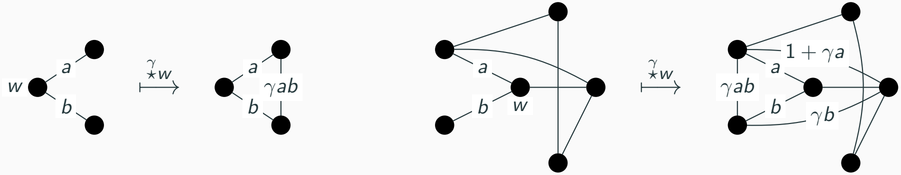


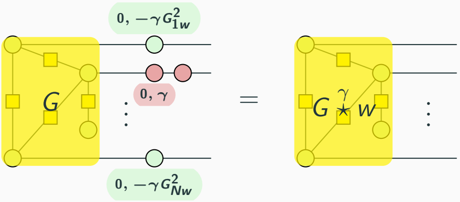
Local scaling



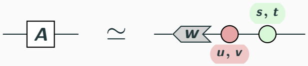


Local complementation





If A is a single-qupit Clifford diagram, then



or

for some $s, t, u, v \in \mathbb{Z}_p$ and $w \in \mathbb{Z}_p^*$. Furthermore, this form is unique.

Completeness (sketch)

1. Turn the diagram $m \to n$ into a state $0 \to m + n$ using cups.

2. Every state in the calculus can be rewritten to GS+LC form.

3. By the previous slide, we can normalise the LC parts, and pull as much of them as possible into local operations on the GS part.

4. The resulting diagram is called reduced GS+LC form, and we know how to compare two diagrams which are in this form.

Mixed states and the CPM construction

- Straightforward extension to a complete axiomatisation for mixed stabiliser states using the discard construction.
- An alternative interpretation in terms of affine co-isotropic relations over $\mathbb{Z}_p$, which is sound and complete.

Titouan Carette et al. "Completeness of Graphical Languages for Mixed State Quantum Mechanics". In: *ACM Transactions on Quantum Computing* 2.4 (Dec. 21, 2021), 17:1–17:28.

Cole Comfort and Aleks Kissinger. "A Graphical Calculus for Lagrangian Relations". May 13, 2021.

Future work

- Cleaner/simplified axiomatisation
  - Can probably eliminate several axioms
  - Phase polynomial normal form?

- Complete ZX-calculi for all of quantum theory in prime dimensions?
  - the unitaries here are parametrised by polynomials: $\omega^{p(x)}$ where $p(x) = ax + bx^2$; take higher orders?
  - ZH_p-calculi more tractable?

- Complete ZX-calculi for stabiliser theory in non-prime dimensions?
  - some representation theory to take advantage of?
  - phase polynomials more tractable?

- What about affine co-isotropic relations over other fields?

Thanks for listening!