Extracting reversible quantum circuits from measurement-based quantum computations with qudits

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Quantum computing with qudits: states

- State space $\to$ d-dimensional Hilbert space:

$$\mathcal{H} = \mathrm{L}^2(\mathbb{Z}_d) \cong \mathbb{C}^d$$

$$|\psi\rangle \in \mathcal{H} \iff |\psi\rangle = \sum_{x \in \mathbb{Z}_d} c_x |x\rangle$$

- d is prime $\implies$ $\mathbb{Z}_d$ is a field

Quantum computing with qudits: gates

- Pauli gates

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- These gates are not self-inverse

$$(X^a Z^b)^d = 1_{\mathcal{H}} \qquad H^4 = 1_{\mathcal{H}} \qquad E^d = 1_{\mathcal{H}\otimes\mathcal{H}}.$$

Measurement patterns

$$\mathfrak{P} \simeq \left(\prod_{v \in O^c}^{\prec} X_{\mathbf{x}(v)}^{m_v} Z_{\mathbf{z}(v)}^{m_v} M_v^{a_v, b_v}(\vec{\theta}_v) \right) \left(\prod_{(u,v) \in G} E_{u,v}^{G_{uv}} \right) \left(\prod_{v \in I^c} N_v \right)$$

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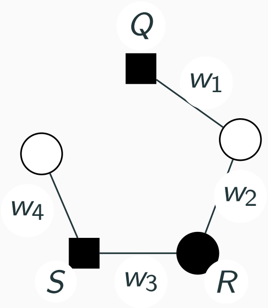

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Past results: necessary and sufficient

- $\mathbb{Z}_d$-flow is a linear-algebraic property that relates the adjacency matrix of the open graph A_G to a matrix describe the corrections.
- $\mathbb{Z}_d$-flow is both necessary and sufficient for robust determinism.
- Robustly deterministic MBQCs implement isometries... but how to find a corresponding *reversible* circuit?

Difficulties

- we know how to extract gate teleportations:

$$\blacksquare\!\!\longrightarrow\!\!\!\!\!-\!\bigcirc = -\boxed{U}-\boxed{H}- = -\boxed{J}- \tag{1}$$

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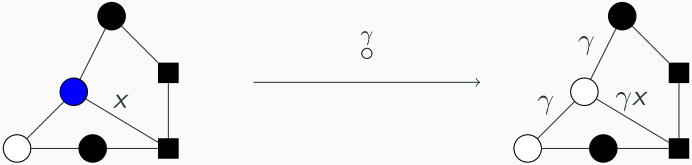
The solution: local Clifford operations

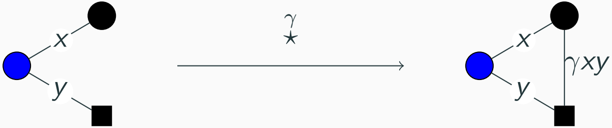
Local scaling



$$G \overset{\gamma}{\circ} w = D^w_\gamma G D^w_\gamma \quad \text{and} \quad C^{\overset{\gamma}{\circ} w} = D^w_{\gamma^{-1}} C$$

Local complementation

$$G \overset{\gamma}{\star} w := G + \gamma G 1_{\{w\}} 1_{\{w\}}^{\mathsf{T}} G - \gamma D^2_{Ge_w}.$$



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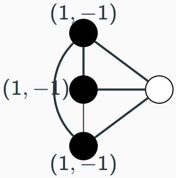
- these two operations preserve the existence of $\mathbb{Z}_d$-flow;
- they preserve the semantics of the pattern (up to updating the choice of measurements);
- they act simultaneously on both the connectivity of the open graph and its labelling;
- this action depends on the connectivity itself...

The extraction algorithm (informally)

1. Using local Cliffords, we can reduce the graph, so that all labels are either $(0,1)$ or $(1,0)$; i.e. measurements are $\mathcal{M}(Z)$ or $\mathcal{M}(X)$;
2. any vertices connected to the outputs and measured in $\mathcal{M}(Z)$ can be extracted using teleportation;
3. if the only vertices connected to outputs are measured in $\mathcal{M}(X)$, then there is a special procedure to map them to $\mathcal{M}(Z)$.

Circuit extraction: an example



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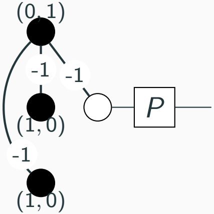

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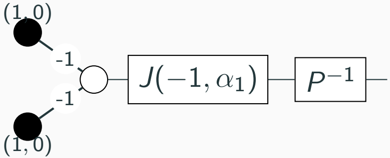

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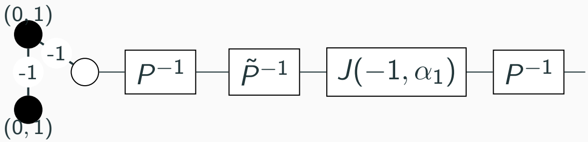

Circuit extraction: an example

$$-\boxed{J(-1,\alpha_3)}-\boxed{J(-1,\alpha_2)}-\boxed{P^{-1}}-\boxed{\tilde{P}^{-1}}-\boxed{J(-1,\alpha_1)}-\boxed{P^{-1}}-$$

Future work

1. improvements to the extraction algorithm;
2. Pauli $\mathbb{F}$-flow?
3. extraction of qudit ZX-calculus diagrams

Thanks for listening!