Complete ZX-calculi for the stabiliser fragment in odd prime dimensions

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Quantum computing with qupits: states

- State space $\rightarrow$ p-dimensional Hilbert space:

$$\mathcal{H} = \mathrm{L}^2(\mathbb{Z}_p) \cong \mathbb{C}^d$$

$$|\psi\rangle \in \mathcal{H} \iff |\psi\rangle = \sum_{x \in \mathbb{Z}_p} c_x |x\rangle$$

- p is prime $\implies$ $\mathbb{Z}_p$ is a field

Quantum computing with qupits: gates

- Pauli gates

$$Z|x\rangle = e^{\frac{i2\pi x}{p}}|x\rangle = \omega^x|x\rangle \quad X|x\rangle = |x+1 \bmod p\rangle$$

$$\text{generic Pauli} \quad \rightsquigarrow \quad X^a Z^b \quad \text{for} \quad a, b \in \mathbb{Z}_p$$

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- Hadamard gate

$$H|x\rangle = \sum_{k\in\mathbb{Z}_p} \omega^{kx}|x\rangle \quad \text{and} \quad HZH^\dagger = X$$

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- Controlled phase or CZ-gate

$$E|x\rangle \otimes |y\rangle = \omega^{xy}|x\rangle \otimes |y\rangle$$

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- Phase gate

$$S|x\rangle = \omega^{2^{-1}x(x+1)}|x\rangle$$

Quantum computing with qupits: gates (cont.)

These gates are not self-inverse

$$(X^a Z^b)^p = 1_{\mathcal{H}} \qquad S^p = 1_{\mathcal{H}} \qquad E^p = 1_{\mathcal{H}\otimes\mathcal{H}} \qquad H^4 = 1_{\mathcal{H}}.$$

The stabiliser fragment

- Recall the qubit stabiliser fragment:

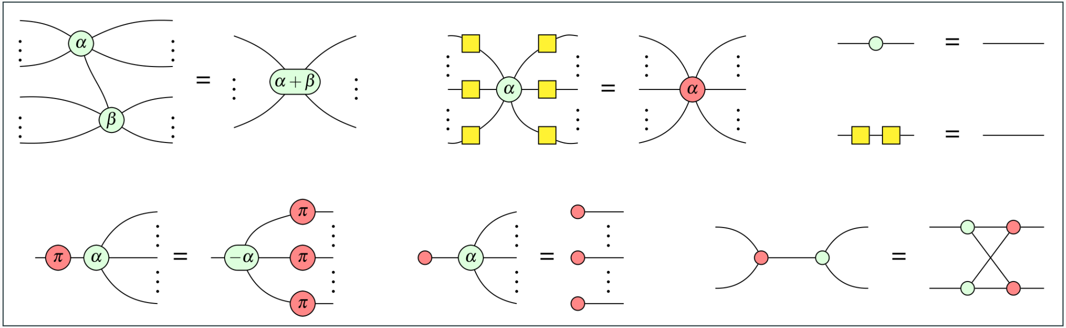
  Pauli states + Clifford unitaries + Pauli projections

- The Clifford group on n qupits is the normaliser in $U(\mathcal{H}^{\otimes n})$ of the group of Paulis.

- Formally, the Clifford group is a representation of $\mathrm{ASymp}(\mathbb{Z}_p^{2n}) \rtimes \mathbb{T}$

  Symplectic matrix + Translation + Global phase

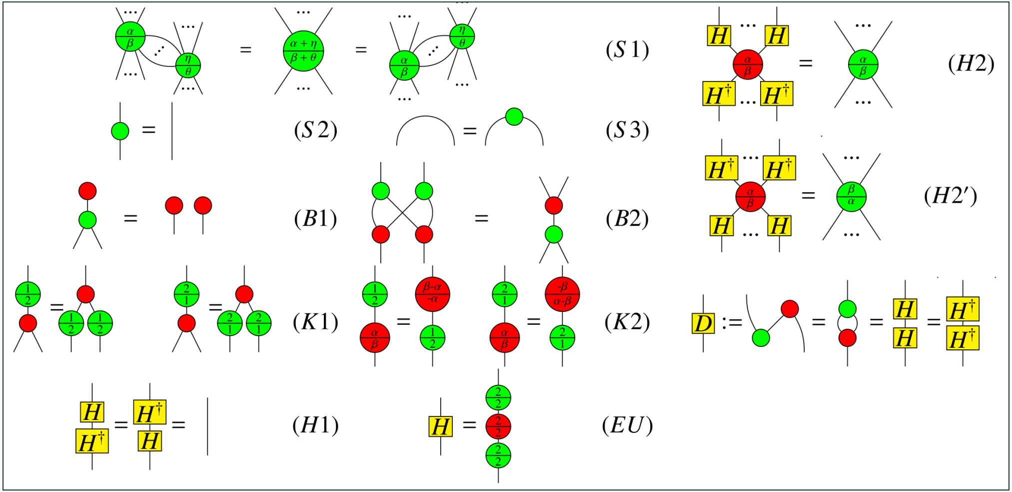
Qubit stabiliser ZX



Miriam Backens. "The ZX-calculus Is Complete for Stabilizer Quantum Mechanics". In: *New Journal of Physics* 16.9 (Sept. 17, 2014), p. 093021.

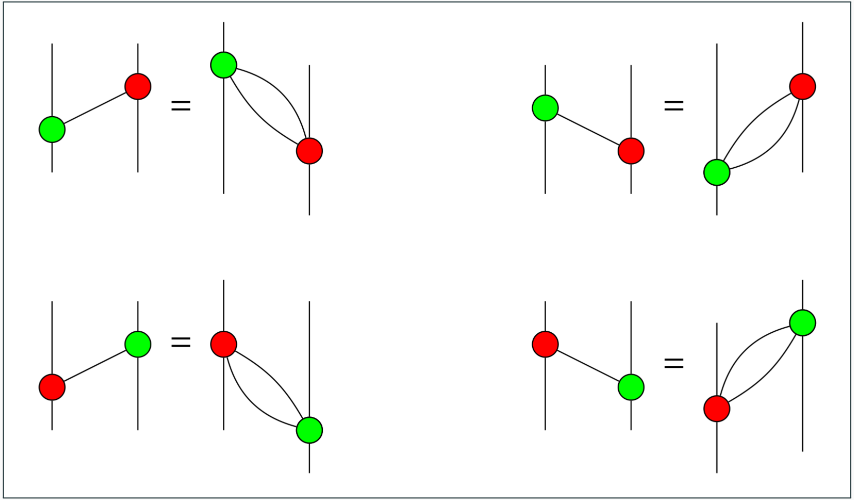
Tommy McElvanney and Miriam Backens. *Complete Flow-Preserving Rewrite Rules for MBQC Patterns with Pauli Measurements*. May 4, 2022.

Qutrit stabiliser ZX



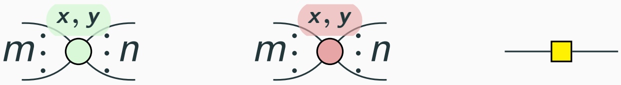
Quanlong Wang. "Completeness of the ZX-calculus". PhD thesis. Oxford: Oxford University, 2018.

Qutrit stabiliser ZX



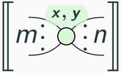
Quanlong Wang. "Completeness of the ZX-calculus". PhD thesis. Oxford: Oxford University, 2018.

Generators of the calculus

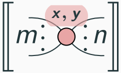


...for $m, n \in \mathbb{N}$ and $x, y \in \mathbb{Z}_p$.

Sequential and parallel composition defined in the usual way.

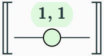


$$= \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \left|k : Z\right\rangle^{\otimes n} \left\langle k : Z\right|^{\otimes m}$$

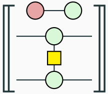


$$= \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \left|-k : X\right\rangle^{\otimes n} \left\langle k : X\right|^{\otimes m}$$

$= H$

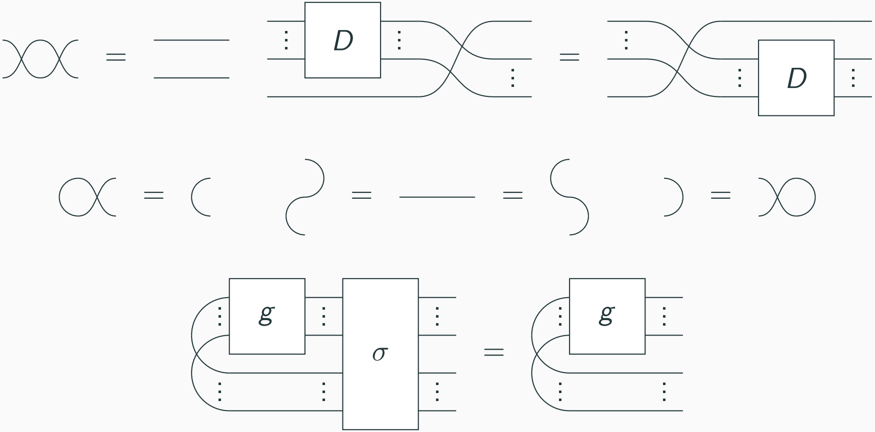


$= S$



$= E$

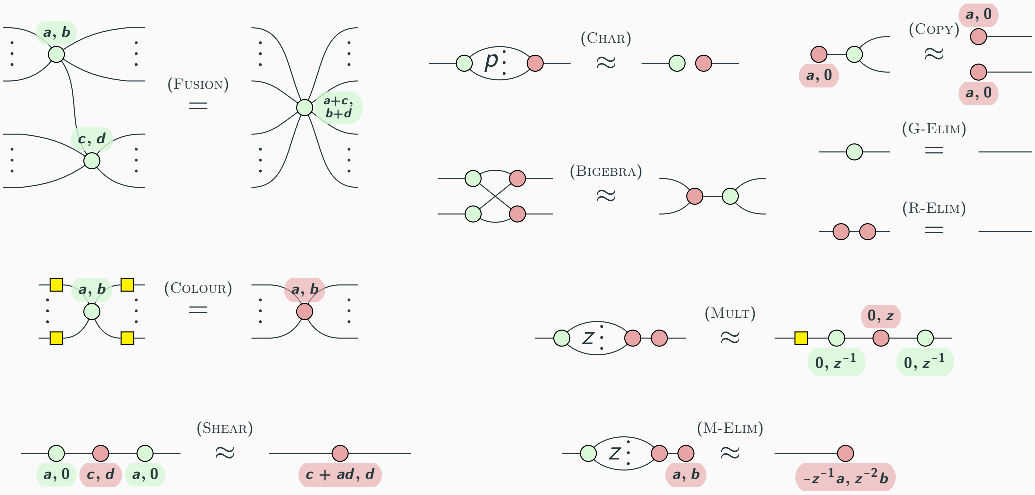
Structural axioms



The calculus is flexsymmetric!

Titouan Carette. “Wielding the ZX-calculus, Flexsymmetry, Mixed States, and Scalable Notations”. Theses. Université de Lorraine, Nov. 2021.

Scalarless axiomatisation



Main results

This calculus is sound, universal and complete for the stabiliser fragment in any odd prime dimension.

Things I don't have time to talk about

- Meta-rule for "spider wars"

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    - Local complementation
- Completeness of the scalar fragment

Mixed states and the CPM construction

- Straightforward extension to a complete axiomatisation for mixed stabiliser states using the discard construction.

- An alternative interpretation in terms of affine co-isotropic relations over $\mathbb{Z}_p$, which is sound and complete:

$$\begin{tikzcd}
\mathsf{ZX}_p \arrow[rr, dashed, "{[-]}"] \arrow[d, "{[\![-]\!]}"'] & & \mathsf{AffCoIsoRel}_{\mathbb{Z}_p} \arrow[d, leftrightarrow] \\
\mathrm{CPM}(\mathsf{Stab}_p) \arrow[rr, two heads] & & \mathrm{CPM}(\mathsf{Stab}_p)/(\textsc{Scalars})
\end{tikzcd}$$

Titouan Carette et al. "Completeness of Graphical Languages for Mixed State Quantum Mechanics". In: *ACM Transactions on Quantum Computing* 2.4 (Dec. 21, 2021), 17:1–17:28.

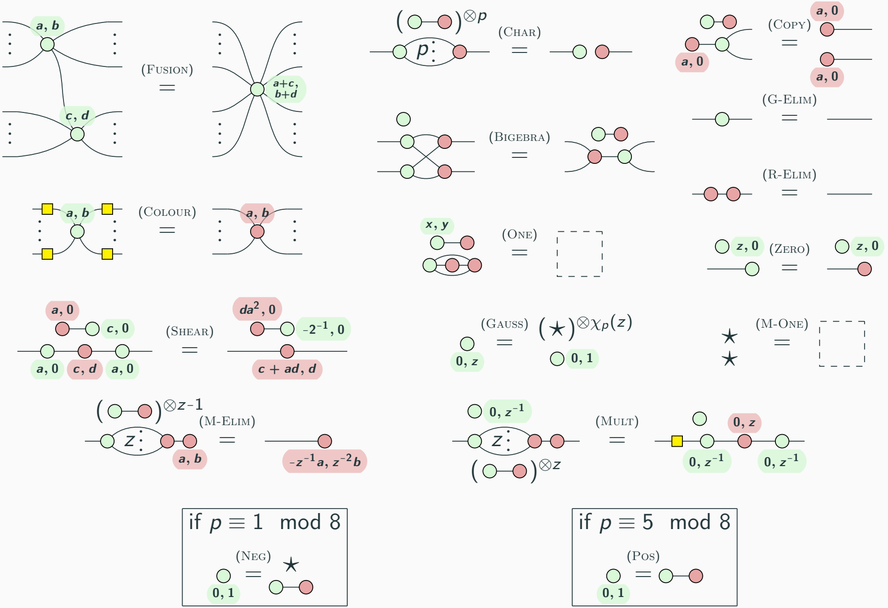
Cole Comfort and Aleks Kissinger. "A Graphical Calculus for Lagrangian Relations". May 13, 2021.

Future work

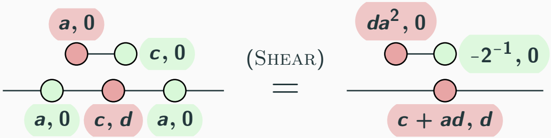
- Cleaner/simplified axiomatisation
- Complete ZX-calculi for all of quantum theory in prime dimensions?
- Complete ZX-calculi for stabiliser theory in non-prime dimensions?
- What about affine co-isotropic relations over other fields?

Thanks for listening!

Full axiomatisation

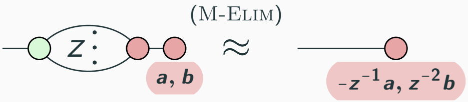


Axiom for affine shears

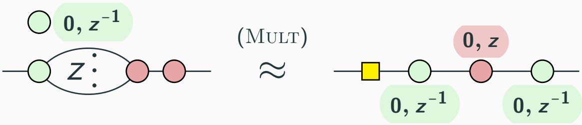


$$\begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ a \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} ad \\ a \end{bmatrix}$$

Axioms for multipliers

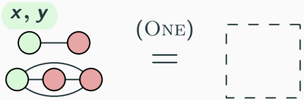


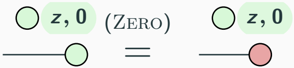
$$|k\rangle \mapsto |zk\rangle$$

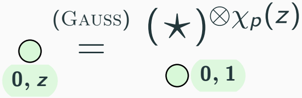


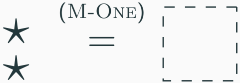
$$\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -z & -1 \end{bmatrix} \begin{bmatrix} 1 & z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Scalar axioms

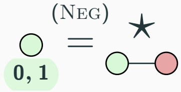




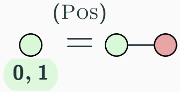




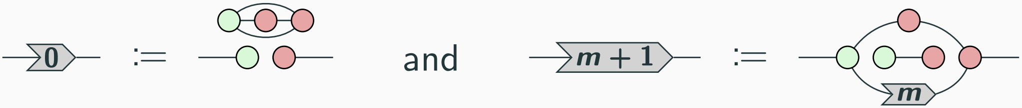
$\text{if } p \equiv 1 \mod 8$

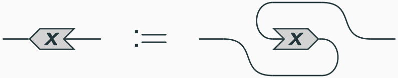


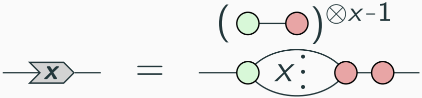
$\text{if } p \equiv 5 \mod 8$



Syntactic sugar for multi-edges



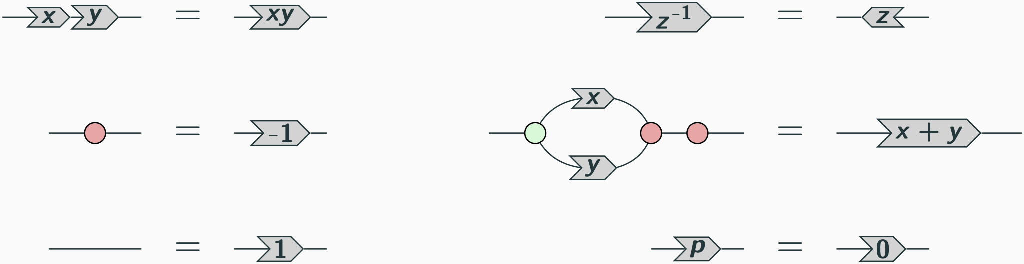




Multipliers are not flexsymmetric!

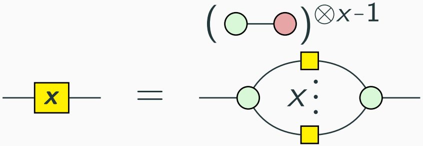
Fabio Zanasi. “Interacting Hopf Algebras: The Theory of Linear Systems”. PhD thesis. Ecole Normale Superieure de Lyon, May 4, 2018.

Syntactic sugar for multi-edges (cont.)



Multipliers are a presentation of the field $\mathbb{Z}_p$!

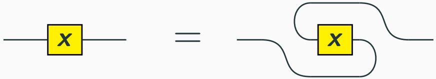
Weighted Hadamards

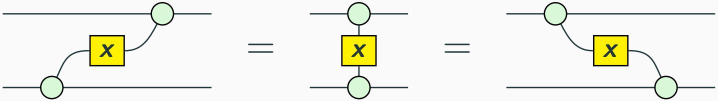


Alex Townsend-Teague and Konstantinos Meichanetzidis. “Classifying Complexity with the ZX-Calculus: Jones Polynomials and Potts Partition Functions”. Mar. 11, 2021.

Weighted Hadamards (cont.)

Weighted Hadamards are flexsymmetric:





Weighted Hadamard equations