Complete ZX-calculi for the stabiliser fragment in odd prime dimensions

Robert I. Booth, Titouan Carette

MFCS 2022

A crash course on quantum theory

- States are vectors from finite-dimensional vector space: $\mathcal{H} = \mathbb{C}^p$

$$|\psi\rangle \in \mathcal{H} \iff |\psi\rangle = \sum_{x \in \mathbb{Z}_p} c_x |x\rangle$$

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- p is prime $\implies$ $\mathbb{Z}_p$ is a field.

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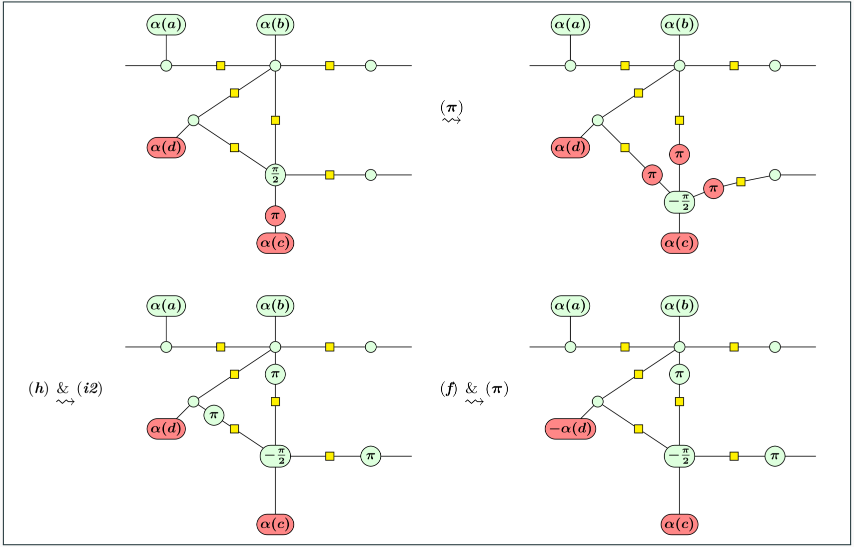
The ZX-calculus



Miriam Backens et al. "There and Back Again: A Circuit Extraction Tale". In: *Quantum* 5 (Mar. 25, 2021), p. 421.

Applications

- Optimisation and compilation of quantum circuits:
    - Ross Duncan et al. "Graph-Theoretic Simplification of Quantum Circuits with the ZX-calculus". In: *Quantum* 4 (June 4, 2020), p. 279
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- ...and many more: `zxcalculus.com` !

The stabiliser fragment

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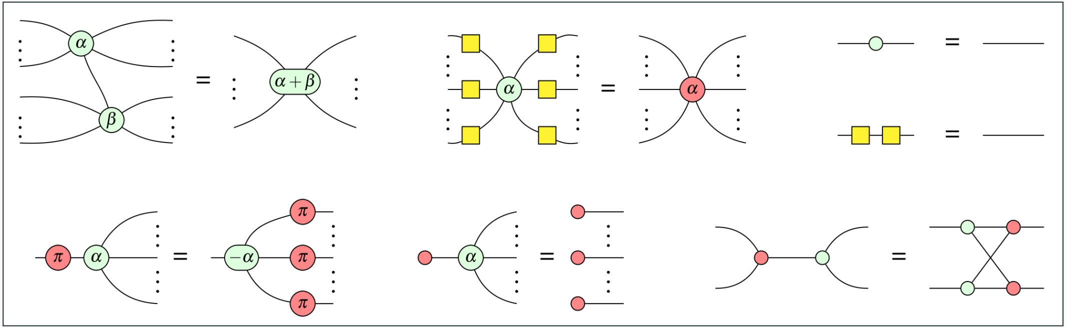
- In odd prime dimensions, the Clifford group is a representation of $\mathrm{ASymp}(\mathbb{Z}_p^{2n}) \rtimes \mathbb{T}$

$$\text{Symplectic matrix} + \text{Translation} + \text{Global phase}$$

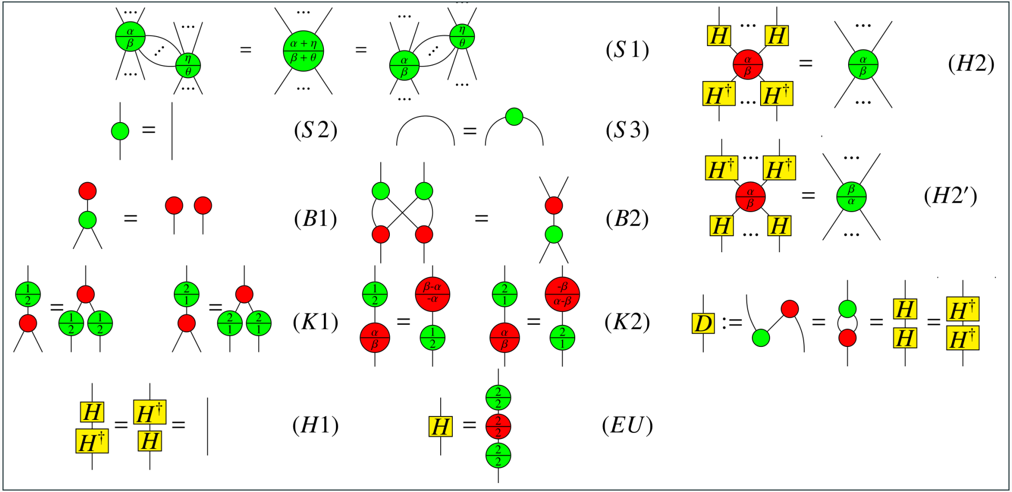
Qubit stabiliser ZX



Miriam Backens. “The ZX-calculus Is Complete for Stabilizer Quantum Mechanics”. In: *New Journal of Physics* 16.9 (Sept. 17, 2014), p. 093021.

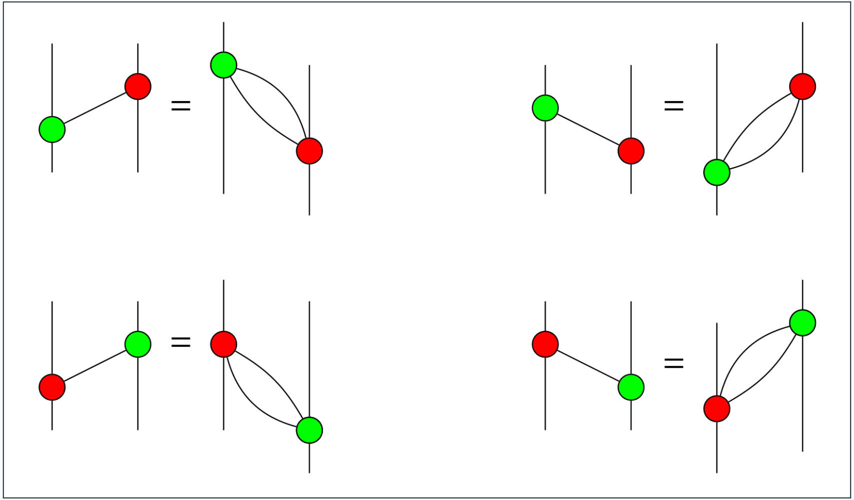
Tommy McElvanney and Miriam Backens. *Complete Flow-Preserving Rewrite Rules for MBQC Patterns with Pauli Measurements*. May 4, 2022.

Qutrit stabiliser ZX



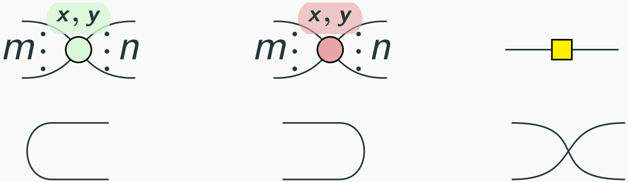
Quanlong Wang. "Completeness of the ZX-calculus". PhD thesis. Oxford: Oxford University, 2018.

Qutrit stabiliser ZX



Quanlong Wang. "Completeness of the ZX-calculus". PhD thesis. Oxford: Oxford University, 2018.

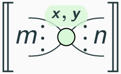
Generators of the calculus



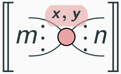
...for $m, n \in \mathbb{N}$ and $x, y \in \mathbb{Z}_p$.

Sequential composition is given by connecting output wires of one diagram to the input wires of another, and parallel composition by vertical juxtaposition of diagrams.

Standard interpretation

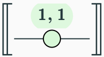


$$= \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \, |k : Z\rangle^{\otimes n} \langle k : Z|^{\otimes m}$$

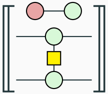


$$= \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \, |-k : X\rangle^{\otimes n} \langle k : X|^{\otimes m}$$

 $\rightsquigarrow$ Fourier gate

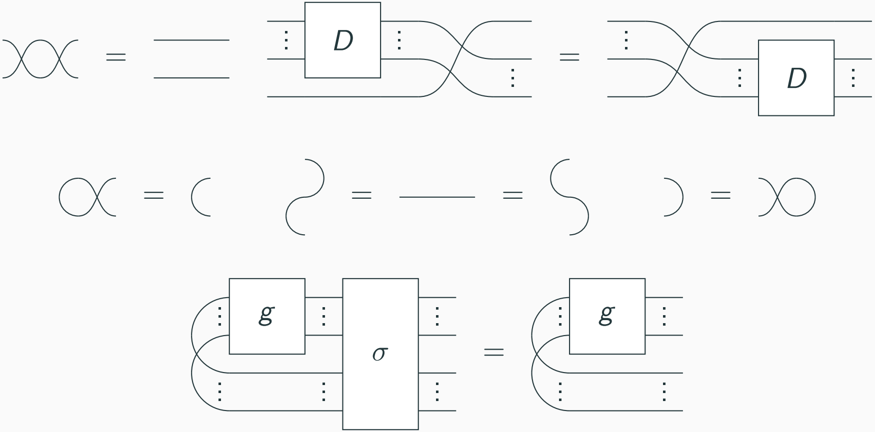


$\rightsquigarrow$ Phase gate

 $\rightsquigarrow$ Controlled-Z gate

The calculus is universal for the stabiliser fragment in dimension p.

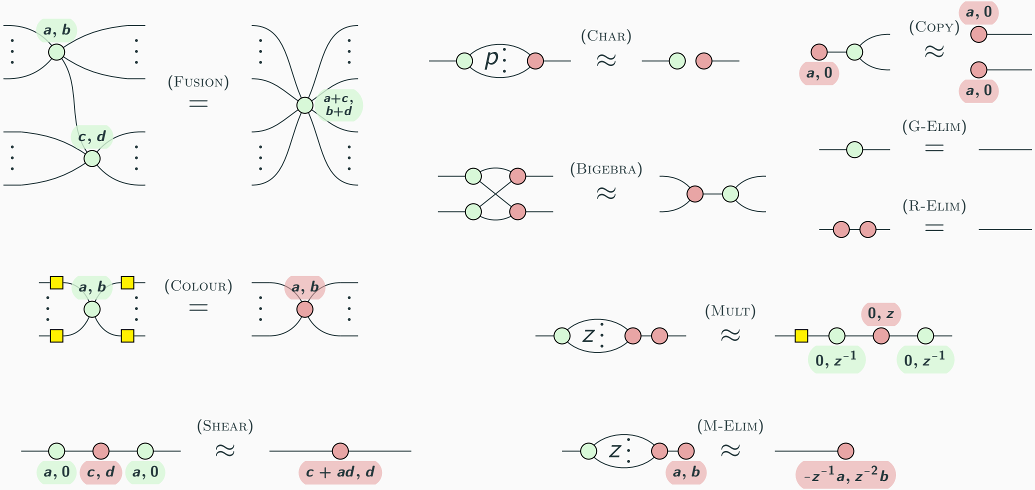
Structural axioms



The calculus is flexsymmetric!

Titouan Carette. "Wielding the ZX-calculus, Flexsymmetry, Mixed States, and Scalable Notations". Theses. Université de Lorraine, Nov. 2021.

Scalarless axiomatisation



Main results

This calculus is sound and complete for the stabiliser fragment in any odd prime dimension.

...and a bunch more things I don't have time to talk about.

Future work

- Cleaner/simplified axiomatisation and normal form

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- Complete ZX-calculi for all of quantum theory in prime dimensions?

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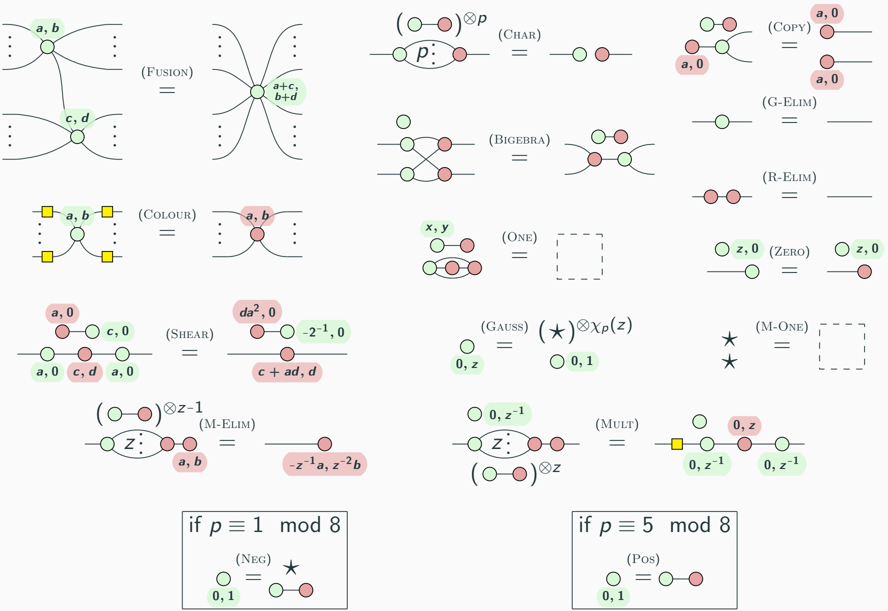
- Complete ZX-calculi for stabiliser theory in non-prime dimensions?

Thanks for listening!

Full axiomatisation



Mixed states and the CPM construction

- Straightforward extension to a complete axiomatisation for mixed stabiliser states using the discard construction.

Titouan Carette et al. "Completeness of Graphical Languages for Mixed State Quantum Mechanics". In: *ACM Transactions on Quantum Computing* 2.4 (Dec. 21, 2021), 17:1–17:28.

Cole Comfort and Aleks Kissinger. "A Graphical Calculus for Lagrangian Relations". May 13, 2021.

Mixed states and the CPM construction

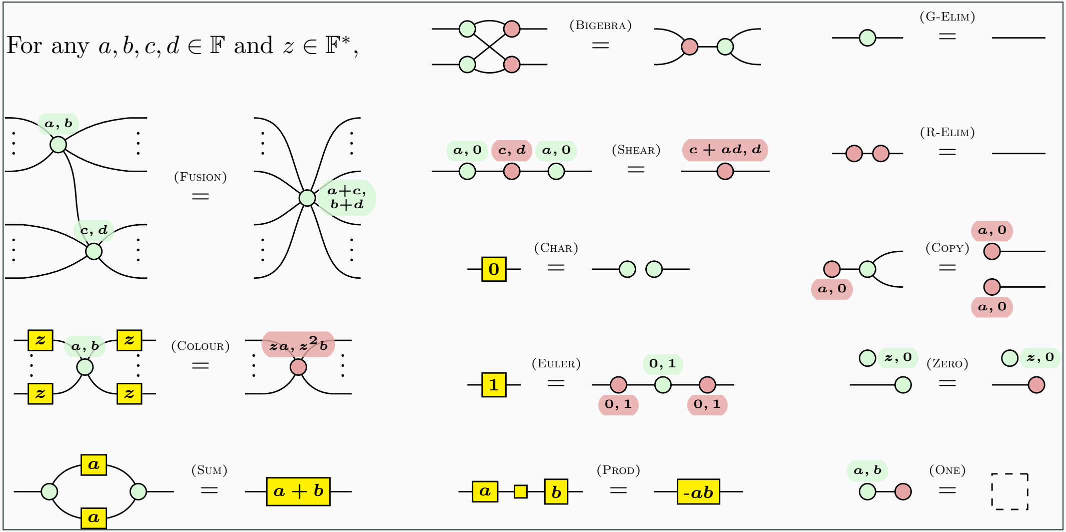
- Straightforward extension to a complete axiomatisation for mixed stabiliser states using the discard construction.
- An alternative interpretation in terms of affine co-isotropic relations over $\mathbb{Z}_p$, which is sound and complete:

$$\begin{tikzcd}
\mathsf{ZX}_p \arrow[r, dashed, "{[-]}"] \arrow[d, "{[\![-]\!]}"'] & \mathsf{AffCoIsoRel}_{\mathbb{Z}_p} \arrow[d, leftrightarrow] \\
\mathrm{CPM}(\mathsf{Stab}_p) \arrow[r, two heads] & \mathrm{CPM}(\mathsf{Stab}_p)/(\textsc{Scalars})
\end{tikzcd}$$

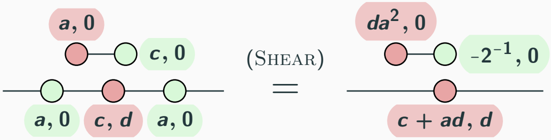
Titouan Carette et al. “Completeness of Graphical Languages for Mixed State Quantum Mechanics”. In: *ACM Transactions on Quantum Computing* 2.4 (Dec. 21, 2021), 17:1–17:28.
Cole Comfort and Aleks Kissinger. “A Graphical Calculus for Lagrangian Relations”. May 13, 2021.

An axiomatisation for affine Lagrangian relations over any field

For any $a, b, c, d \in \mathbb{F}$ and $z \in \mathbb{F}^*$,

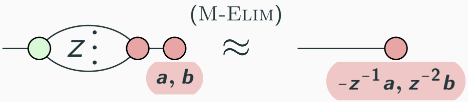


Axiom for affine shears

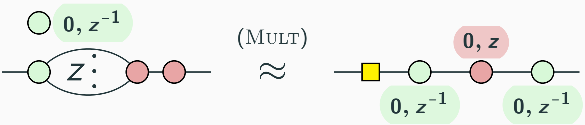


$$\begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ a \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} ad \\ a \end{bmatrix}$$

Axioms for multipliers

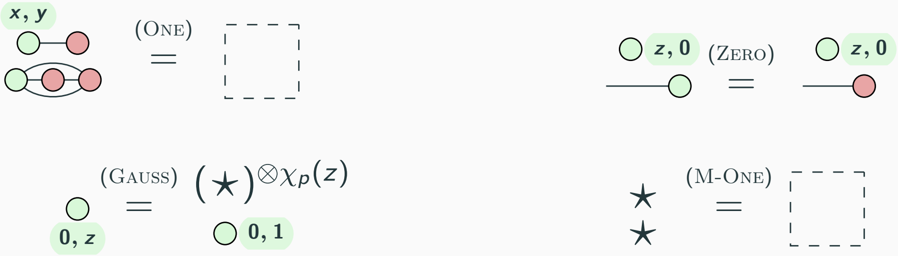


$$|k\rangle \mapsto |zk\rangle$$

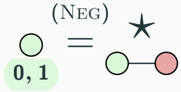


$$\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -z & -1 \end{bmatrix} \begin{bmatrix} 1 & z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

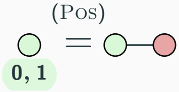
Scalar axioms



if $p \equiv 1 \mod 8$

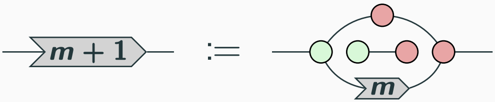


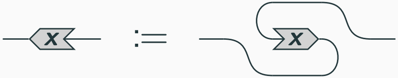
if $p \equiv 5 \mod 8$

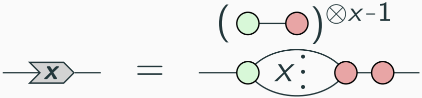


Syntactic sugar for multi-edges

and



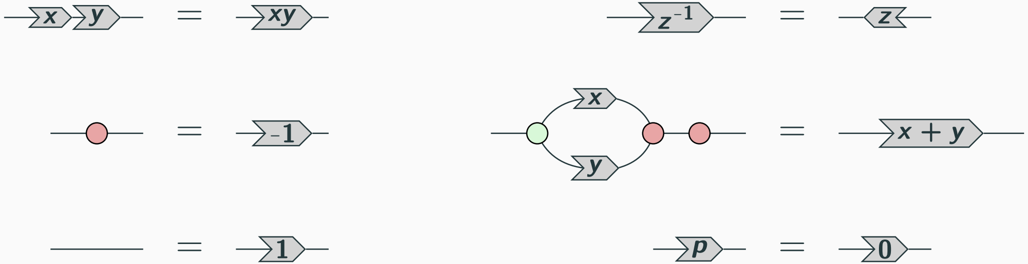




Multipliers are not flexsymmetric!

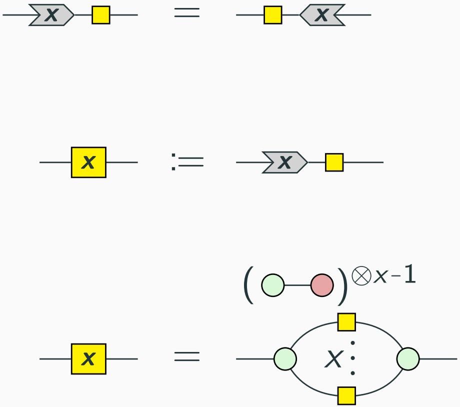
Fabio Zanasi. "Interacting Hopf Algebras: The Theory of Linear Systems". PhD thesis. Ecole Normale Superieure de Lyon, May 4, 2018.

Syntactic sugar for multi-edges (cont.)



Multipliers are a presentation of the field $\mathbb{Z}_p$!

Weighted Hadamards



Alex Townsend-Teague and Konstantinos Meichanetzidis. “Classifying Complexity with the ZX-Calculus: Jones Polynomials and Potts Partition Functions”. Mar. 11, 2021.

Weighted Hadamards (cont.)

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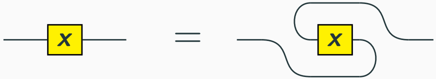

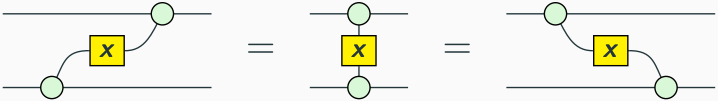

Weighted Hadamard equations