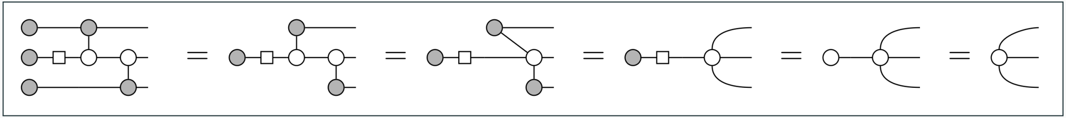
Diagrammatic reasoning: past, present, ... future?

Robert I. Booth

LIP6 QI Group Meeting, 29 June 2023

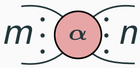
Part one: the Past

Diagrammatic reasoning in a nutshell

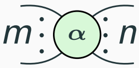


John van de Wetering. **"ZX-calculus for the Working Quantum Computer Scientist".** Dec. 27, 2020.

Generators: spiders and boxes



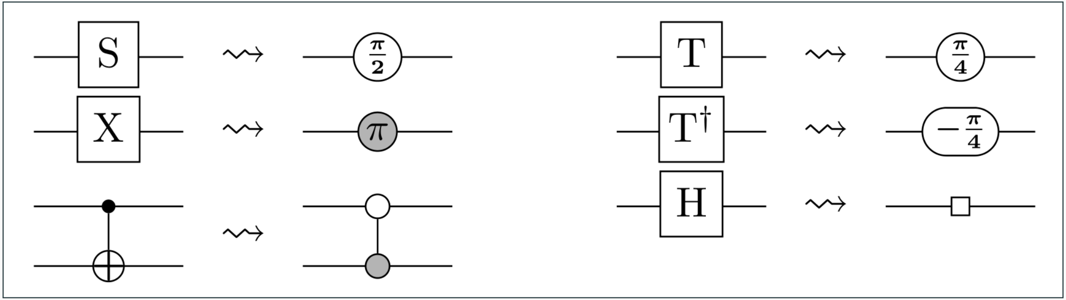
$$= \begin{cases} |0\rangle^{\otimes m} \mapsto |0\rangle^{\otimes n} \\ |1\rangle^{\otimes m} \mapsto e^{i\alpha}\, |1\rangle^{\otimes n} \end{cases}$$



$$= \begin{cases} |+\rangle^{\otimes m} \mapsto |+\rangle^{\otimes n} \\ |-\rangle^{\otimes m} \mapsto e^{i\alpha}\, |-\rangle^{\otimes n} \end{cases}$$

$$= \begin{cases} |0\rangle \mapsto |+\rangle \\ |1\rangle \mapsto |-\rangle \end{cases}$$

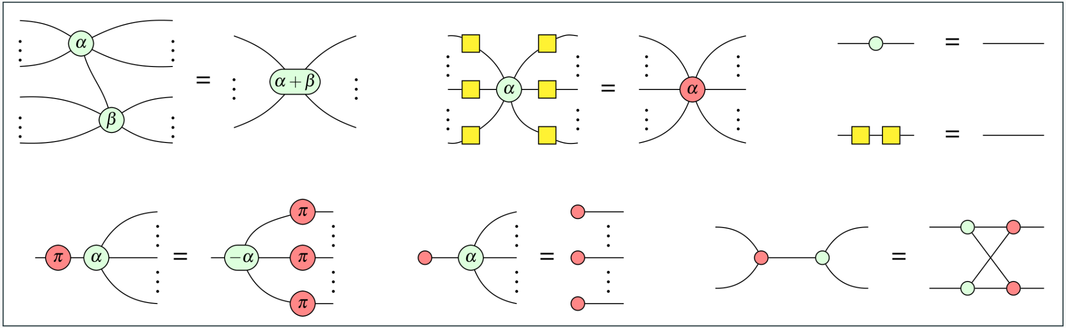
Representation of some common gates



John van de Wetering. **“ZX-calculus for the Working Quantum Computer Scientist”.** Dec. 27, 2020.

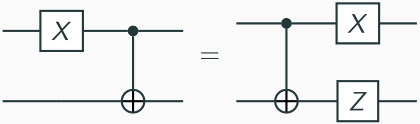
Some rewrite rules

- The quintissential rule: only connectivity matters!

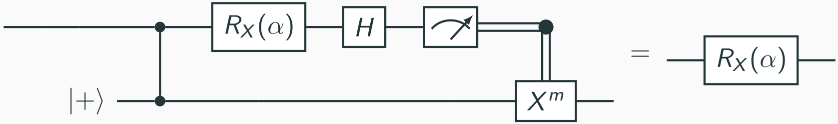


Tommy McElvanney and Miriam Backens. **"Complete Flow-Preserving Rewrite Rules for MBQC Patterns with Pauli Measurements".** In: *QPL 2022* (May 4, 2022).

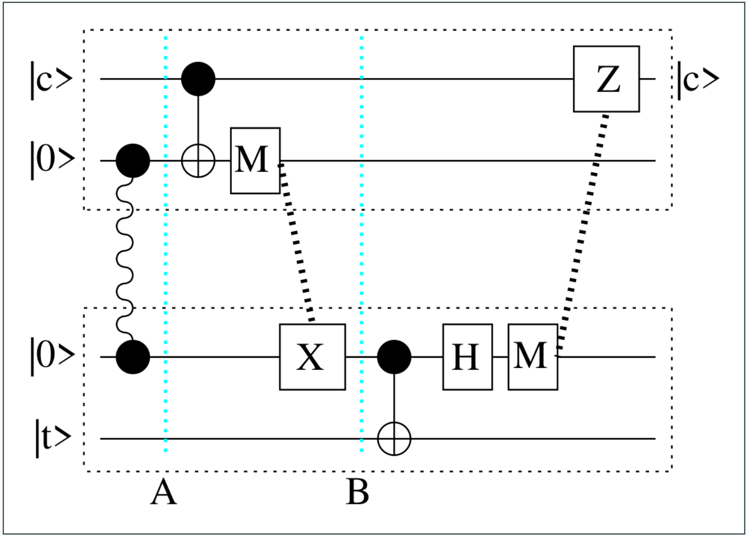
Example 1: Stabilisers of CNOT gate



Example 2: Quantum gate teleportation



Example 3: Cat gadgets



yimsiriwattana˙generalized˙2004.

Caveats

- the ZX-calculus offers no guarantee of realisability/physicality;
- the ZX-calculus is very convenient for reasoning about quantum computation, but might not be so practical for other quantum information questions; other graphical languages exist which might be of more use;
-

Part two: the Present

Quantum computing with qupits: states

- State space $\rightarrow$ p-dimensional Hilbert space:

$$\mathcal{H} = \mathrm{L}^2(\mathbb{Z}_p) \cong \mathbb{C}^d$$

$$|\psi\rangle \in \mathcal{H} \iff |\psi\rangle = \sum_{x \in \mathbb{Z}_p} c_x |x\rangle$$

- p is prime $\implies$ $\mathbb{Z}_p$ is a field

Quantum computing with qupits: gates

- Pauli gates

$$Z|x\rangle = e^{\frac{i2\pi x}{p}}|x\rangle = \omega^x|x\rangle \quad X|x\rangle = |x+1 \bmod p\rangle$$

$$\text{generic Pauli} \quad \rightsquigarrow \quad X^a Z^b \quad \text{for} \quad a, b \in \mathbb{Z}_p$$

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- Hadamard gate

$$H\left|x\right\rangle = \sum_{k\in\mathbb{Z}_p}\omega^{kx}\left|x\right\rangle \quad \text{and} \quad HZH^{\dagger} = X$$

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- Controlled phase or CZ-gate

$$E|x\rangle \otimes |y\rangle = \omega^{xy}|x\rangle \otimes |y\rangle$$

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- Phase gate

$$S|x\rangle = \omega^{2^{-1}x(x+1)}|x\rangle$$

Quantum computing with qupits: gates (cont.)

These gates are not self-inverse

$$(X^a Z^b)^p = 1_{\mathcal{H}} \qquad S^p = 1_{\mathcal{H}} \qquad E^p = 1_{\mathcal{H}\otimes\mathcal{H}} \qquad H^4 = 1_{\mathcal{H}}.$$

The stabiliser fragment

- Recall the qubit stabiliser fragment:

$$\text{Pauli states} + \text{Clifford unitaries} + \text{Pauli projections}$$

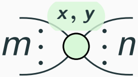
- The Clifford group on n qupits is the normaliser in $U(\mathcal{H}^{\otimes n})$ of the group of Paulis.

$$\text{Symplectic matrix} + \text{Translation} + \text{Global phase}$$

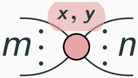
- Formally, the Clifford group is a representation of $\mathrm{ASymp}(\mathbb{Z}_p^{2n}) \rtimes \mathbb{T}$

Generators of the calculus

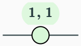
Letting $x, y \in \mathbb{Z}_p$:



$$= |k : Z\rangle^{\otimes m} \mapsto \omega^{2^{-1}(xk+yk^2)} |k : Z\rangle^{\otimes n}$$

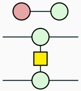


$$= |-k : X\rangle^{\otimes m} \mapsto \omega^{2^{-1}(xk+yk^2)} |k : X\rangle^{\otimes n}$$



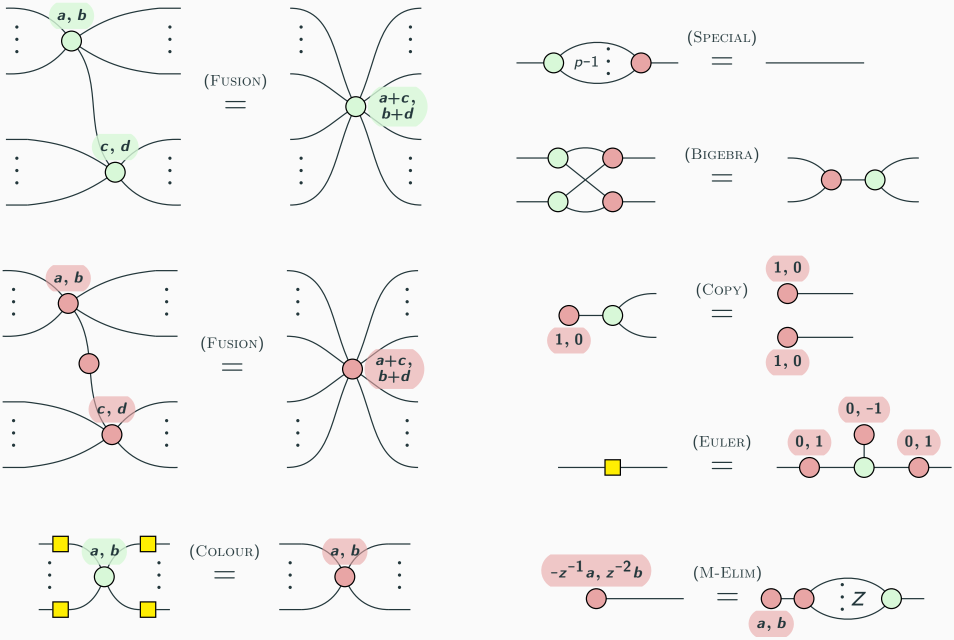
$= S$

$= H$



$= E$

Scalarless axiomatisation



Boring list of results

- The axiomatisation is complete for the Clifford fragment: you can prove anything about Clifford ZX-diagrams using just these 8 rewrite rules;
- In fact, we have a unique normal form for Clifford diagrams;
- We also have a unique normal form for Clifford unitary circuits;
- We also recover a nice representation of graph states as ZX-diagrams, and the action of local-Clifford operations on them.

Robert I. Booth and Titouan Carette. **"Complete ZX-Calculi for the Stabiliser Fragment in Odd Prime Dimensions".** In: *47th International Symposium on Mathematical Foundations of Computer Science (MFCS 2022)*. Ed. by Stefan Szeider, Robert Ganian, and Alexandra Silva. Vol. 241. Leibniz International Proceedings in Informatics (LIPIcs). ISSN: 1868-8969. Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022, 24:1–24:15.

Boldizsár Poór et al. **"The Qupit Stabiliser ZX-travaganza: Simplified Axioms, Normal Forms and Graph-Theoretic Simplification".** In: To Appear in QPL 2023. EPCTS, June 8, 2023.

Part three: the Future?

The symplectic picture of Clifford quantum mechanics

Recall the Canonical Commutation Relations in exponential form:

$$X(\vec{p})Z(\vec{q}) = e^{i2\pi\vec{q}^{\mathsf{T}}\vec{p}}Z(\vec{q})X(\vec{p})$$

which can be reformulated as

$$[X(\vec{p}), Z(\vec{q})] = e^{i2\pi(\vec{q}^{\mathsf{T}}\vec{p}-\vec{p}^{\mathsf{T}}\vec{q})} = e^{i2\pi\begin{pmatrix} q \\ p \end{pmatrix}^{\mathsf{T}}\Omega\begin{pmatrix} q \\ p \end{pmatrix})}$$

where $\Omega = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$

The affine symplectic group

Recall the defining property of the Clifford group:

$$UPU^* = P \quad \text{for any Pauli} \quad P$$

It's clear that all Paulis are in the Clifford group, this gives an affine part. But there are *other* maps in the Clifford group which are not Pauli. We can obtain* these in the form of matrices S such that

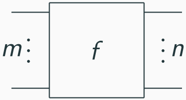
$$S^{\mathsf{T}}\Omega S = \Omega$$

These matrices are called symplectic. In odd prime dimensions, and in CV, *all* Clifford operators arise in this way:

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto S \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} z \\ x \end{pmatrix}$$

The problem, from the perspective of functions

Consider a function $m \to n$, which could be depicted as:



what then are states:

and

i.e. $f : \varnothing \to [1]$ and $g : [1] \to \varnothing$.


MAGIKARP
:L5
HP:
GYARADOS
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HP:
18/ 18
No effect!

Relations

A relation $m \to n$ is a subset of $[m] \times [n]$, and composition is defined by

$$S \circ T = \{(x, z) \mid \exists y \quad \text{such that} \quad (x, y) \in T \quad \text{and} \quad (y, z) \in S\}$$

If $f : [m] \to [n]$ is a function, there is a relation $R_f = \{(x, f(x)) \mid x \in [m]\}$ and relational composition agrees with functional composition.

Now there are nontrivial states and effects:

Affine Lagrangian relations

If M is a matrix $\mathbb{R}^m \to \mathbb{R}^n$, the relation R_M is actually a *subspace* of $\mathbb{R}^m \times \mathbb{R}^n$.

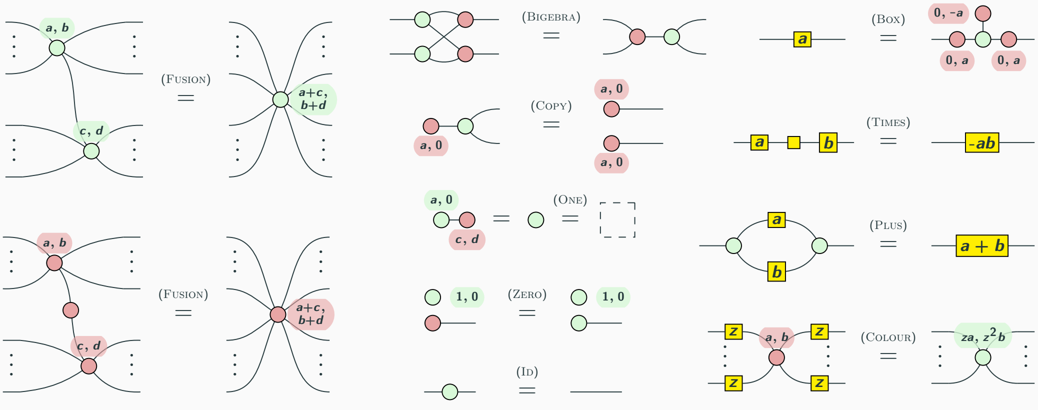
If M is an affine transformation of $\mathbb{R}^m \to \mathbb{R}^n$, the relation R_M is actually an *affine subspace* of $\mathbb{R}^m \times \mathbb{R}^n$.

The relation R_S associated to an affine symplectic map S is a very special kind of subspace, known as an affine Lagrangian relation.

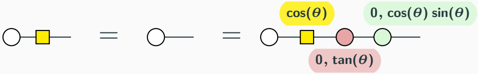
Cole Comfort and Aleks Kissinger. **"A Graphical Calculus for Lagrangian Relations".** May 13, 2021.

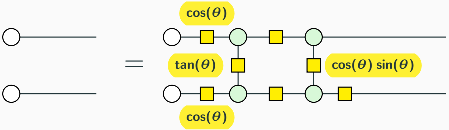
Robert I. Booth and Titouan Carette. **"Complete ZX-Calculi for the Stabiliser Fragment in Odd Prime Dimensions".** In: *47th International Symposium on Mathematical Foundations of Computer Science (MFCS 2022)*. Ed. by Stefan Szeider, Robert Ganian, and Alexandra Silva. Vol. 241. Leibniz International Proceedings in Informatics (LIPIcs). ISSN: 1868-8969. Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022, 24:1–24:15.

Affine Lagrangian relations, diagrammatically



Gaussian quantum mechanics, diagrammatically





Future work

- Can we give a semantics of $\mathsf{AffLagRel}_{\mathbb{R}}$ in some kind of *topological* extension of Hilbert spaces? This is necessary to reintroduce scalars.
- Can we give a satisfying description of taking infinite-squeezing limits?
- Can we extend this presentation to include non-Gaussianity? For instance, can we add Fock states and get a diagrammatic axiomatisation for Fock space?

Advertisements

- Visit our website: `https://zxcalculus.com`
- Connect to our Discord (via the website)
- Come to our weekly meetings (on break until September)

Thanks for listening!