Diagrammatic reasoning beyond qubits

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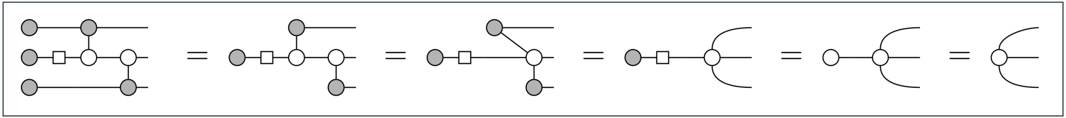
University of BRISTOL

THE UNIVERSITY of EDINBURGH informatics

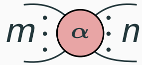
Part one: qubit ZX-calculus

Diagrammatic reasoning in a nutshell

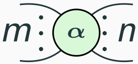


van de Wetering, "ZX-calculus for the Working Quantum Computer Scientist"

Generators: spiders and boxes



$$= \begin{cases} |0\rangle^{\otimes m} \mapsto |0\rangle^{\otimes n} \\ |1\rangle^{\otimes m} \mapsto e^{i\alpha} |1\rangle^{\otimes n} \end{cases}$$



$$= \begin{cases} |+\rangle^{\otimes m} \mapsto |+\rangle^{\otimes n} \\ |-\rangle^{\otimes m} \mapsto e^{i\alpha} |-\rangle^{\otimes n} \end{cases}$$

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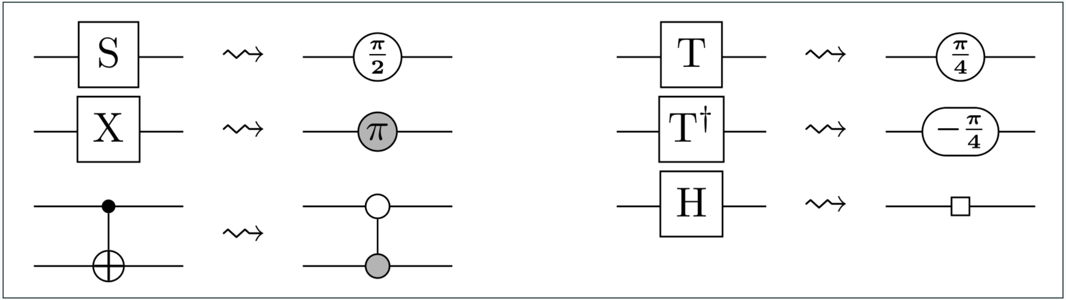

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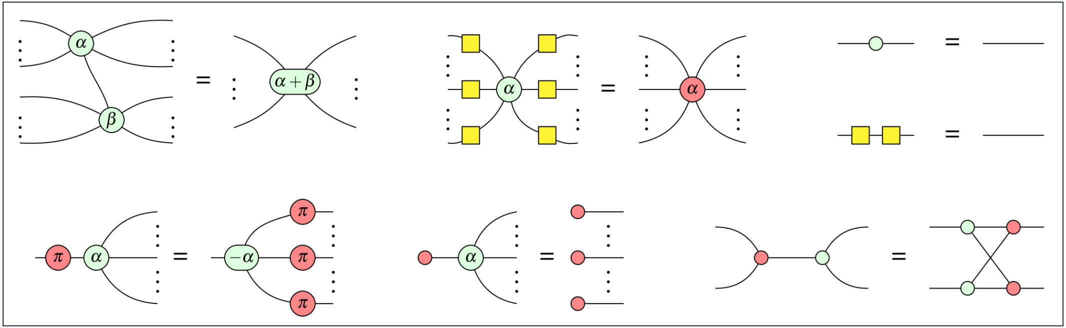
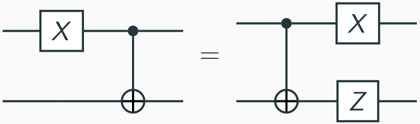
Some rewrite rules

- The quintessential rule: only connectivity matters!

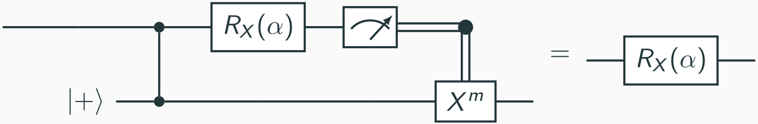


McElvanney and Backens, "Complete Flow-Preserving Rewrite Rules for MBQC Patterns with Pauli Measurements"

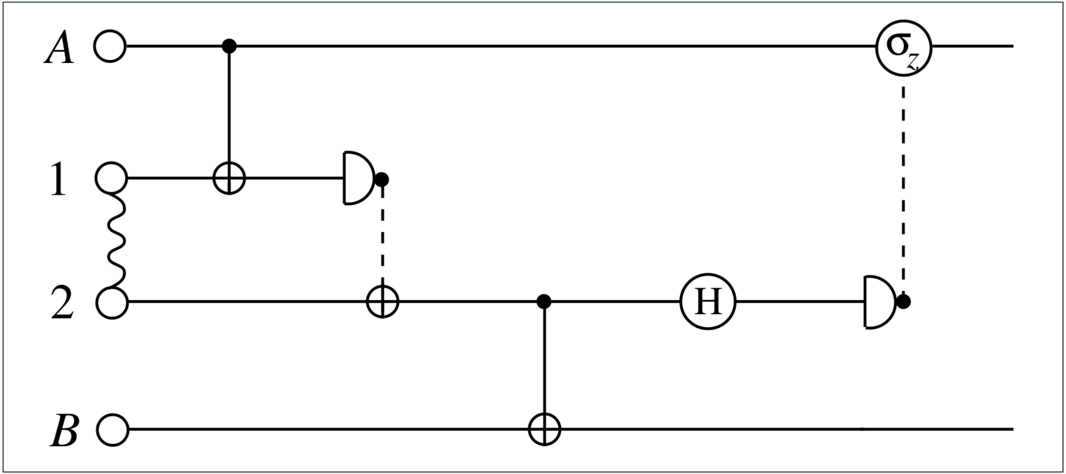
Example 1: Stabilisers of CNOT gate



Example 2: Quantum gate teleportation



Example 3: Cat gadgets

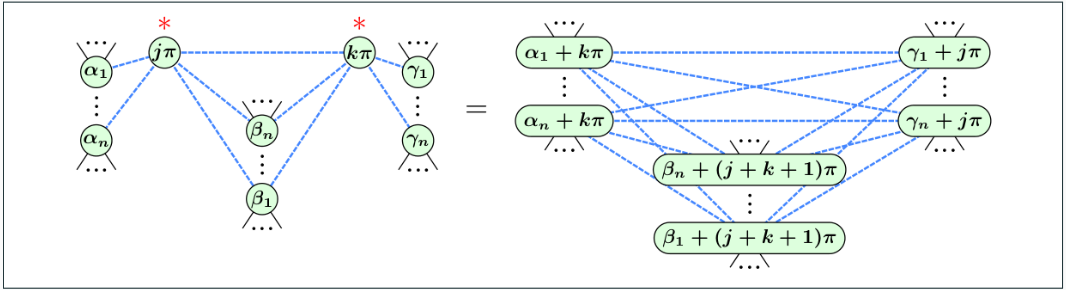


Eisert et al., "Optimal local implementation of non-local quantum gates"

Caveats

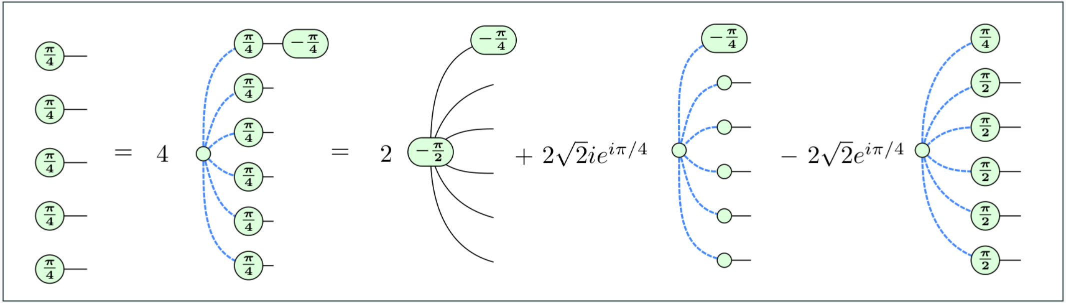
- the ZX-calculus offers no guarantee of realisability/physicality;
- the ZX-calculus is very convenient for reasoning about quantum circuits, but might not be so practical for other quantum information questions; other graphical languages exist which might be of more use;

Application: circuit optimisation



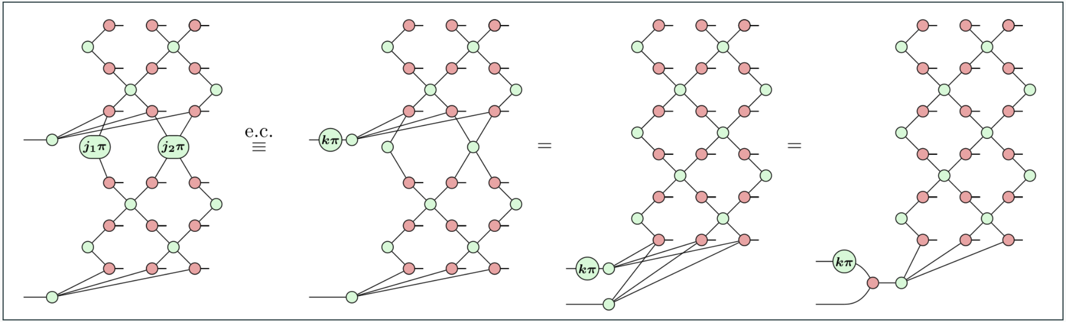
Duncan et al., "Graph-theoretic Simplification of Quantum Circuits with the ZX-calculus"

Application: simulation



Kissinger, Wetering, and Vilmart, "Classical Simulation of Quantum Circuits with Partial and Graphical Stabiliser Decompositions"

Application: error correction



Kissinger, "Phase-free ZX diagrams are CSS codes (...or how to graphically grok the surface code)"

Part two: odd prime dimensions

Quantum computing with quopits: states

- State space $\rightarrow$ p-dimensional Hilbert space:

$$\mathcal{H} = \mathrm{L}^2(\mathbb{Z}_p) \cong \mathbb{C}^d$$

$$|\psi\rangle \in \mathcal{H} \iff |\psi\rangle = \sum_{x \in \mathbb{Z}_p} c_x |x\rangle$$

- p is prime $\implies$ $\mathbb{Z}_p$ is a field

Quantum computing with quopits: gates

- Pauli gates

$$Z|x\rangle = e^{\frac{i2\pi x}{p}}|x\rangle = \omega^x|x\rangle \quad X|x\rangle = |x+1 \bmod p\rangle$$

$$\text{generic Pauli} \quad \rightsquigarrow \quad X^a Z^b \quad \text{for} \quad a, b \in \mathbb{Z}_p$$

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$$E|x\rangle \otimes |y\rangle = \omega^{xy}|x\rangle \otimes |y\rangle$$

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- Phase gate

$$S|x\rangle = \omega^{2^{-1}x(x+1)}|x\rangle$$

Quantum computing with quopits: gates (cont.)

These gates are not self-inverse

$$(X^a Z^b)^p = 1_{\mathcal{H}} \qquad S^p = 1_{\mathcal{H}} \qquad E^p = 1_{\mathcal{H}\otimes\mathcal{H}} \qquad H^4 = 1_{\mathcal{H}}.$$

The stabiliser fragment

- Recall the qubit stabiliser fragment:

  Pauli states + Clifford unitaries + Pauli projections

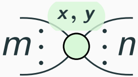
- The Clifford group on n quopits is the normaliser in $U(\mathcal{H}^{\otimes n})$ of the group of Paulis.

  Symplectic matrix + Translation + Global phase

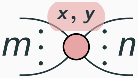
- Formally, the Clifford group is a representation of $\mathrm{ASymp}(\mathbb{Z}_p^{2n}) \rtimes \mathbb{T}$

Generators of the calculus

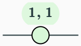
Letting $x, y \in \mathbb{Z}_p$:



$$= |k : Z\rangle^{\otimes m} \mapsto \omega^{2^{-1}(xk+yk^2)} |k : Z\rangle^{\otimes n}$$

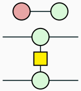


$$= |-k : X\rangle^{\otimes m} \mapsto \omega^{2^{-1}(xk+yk^2)} |k : X\rangle^{\otimes n}$$



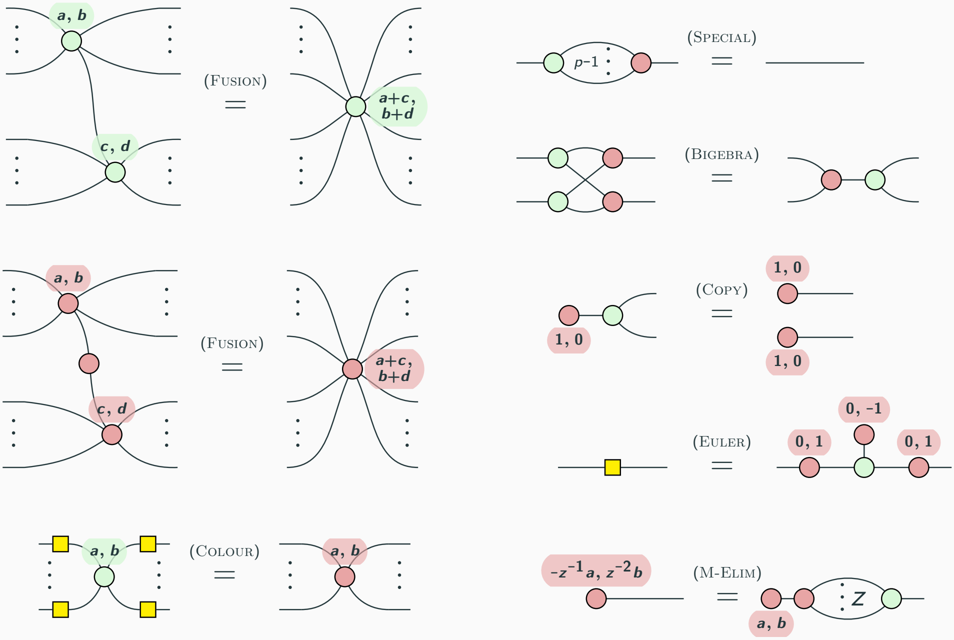
$= S$

$= H$



$= E$

Scalarless axiomatisation



Boring list of results

- The axiomatisation is complete for the Clifford fragment: you can prove anything about Clifford ZX-diagrams using just these 8 rewrite rules;
- In fact, we have a unique normal form for Clifford diagrams;
- We also have a unique normal form for Clifford unitary circuits;
- We also recover a nice representation of graph states as ZX-diagrams, and the action of local-Clifford operations on them.

Booth and Carette, "Complete ZX-Calculi for the Stabiliser Fragment in Odd Prime Dimensions"
Poór et al., "The Qupit Stabiliser ZX-travaganza: Simplified Axioms, Normal Forms and Graph-Theoretic Simplification"

Part three: quantum optics

The symplectic picture of Clifford quantum mechanics

Recall the Canonical Commutation Relations in exponential form:

$$X(\vec{p})Z(\vec{q}) = e^{i2\pi\vec{q}^{\mathsf{T}}\vec{p}}Z(\vec{q})X(\vec{p})$$

which can be reformulated as

$$[X(\vec{p}), Z(\vec{q})] = e^{i2\pi(\vec{q}^{\mathsf{T}}\vec{p}-\vec{p}^{\mathsf{T}}\vec{q})} = e^{i2\pi\begin{pmatrix}q\\p\end{pmatrix}^{\mathsf{T}}\Omega\begin{pmatrix}q\\p\end{pmatrix})}$$

where $\Omega = \begin{pmatrix}0_n & -1_n\\1_n & 0_n\end{pmatrix}$

The affine symplectic group

Recall the defining property of the Clifford group:

$$UPU^* = P \quad \text{for any Pauli} \quad P$$

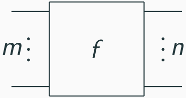
It's clear that all Paulis are in the Clifford group, this gives an affine part. But there are *other* maps in the Clifford group which are not Pauli. We can obtain* these in the form of matrices S such that

$$S^{\mathsf{T}}\Omega S = \Omega$$

These matrices are called symplectic. In odd prime dimensions, and in CV, *all* Clifford operators arise in this way:

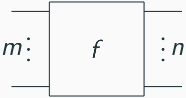
$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto S \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} z \\ x \end{pmatrix}$$

Consider a function $[m] \to [n]$, which could be depicted as:



The problem, from the perspective of functions

Consider a function $[m] \to [n]$, which could be depicted as:



Then *states* are:

and

i.e. $f : \varnothing \to [1]$ and $g : [1] \to \varnothing$.


MAGIKARP
:L5
HP:
GYARADOS
:L5
HP:
18/ 18
No effect!

Relations

A relation $m \to n$ is a subset of $[m] \times [n]$, and composition is defined by

$$S \circ T = \{(x, z) \mid \exists y \quad \text{such that} \quad (x, y) \in T \quad \text{and} \quad (y, z) \in S\}$$

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Now there are nontrivial states and effects:

Affine Lagrangian relations

If M is a matrix $\mathbb{R}^m \to \mathbb{R}^n$, the relation R_M is actually a *subspace* of $\mathbb{R}^m \times \mathbb{R}^n$.

Comfort and Kissinger, "A Graphical Calculus for Lagrangian Relations"
Booth and Carette, "Complete ZX-Calculi for the Stabiliser Fragment in Odd Prime Dimensions"

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The relation R_S associated to an affine symplectic map S is a very special kind of subspace, known as an affine Lagrangian relation.

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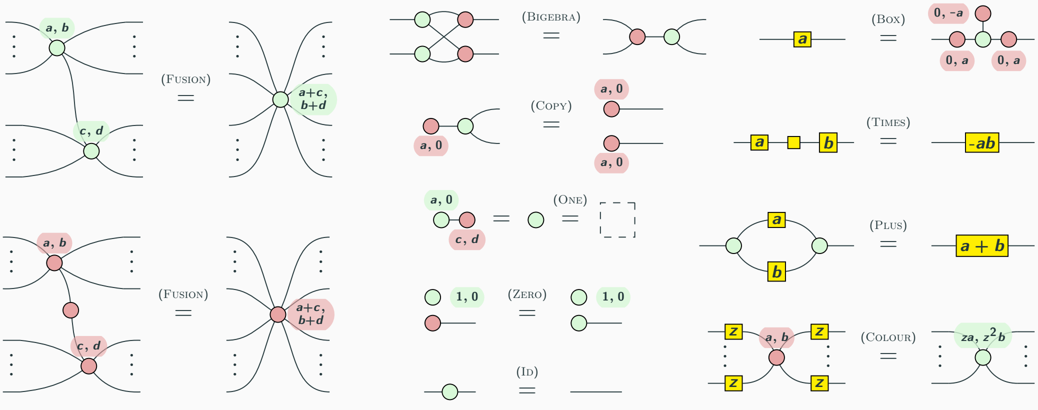
Affine Lagrangian relations, diagrammatically



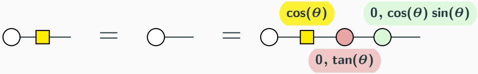
Gaussian quantum mechanics lives in the complexification

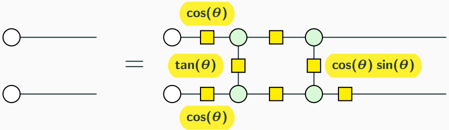
- The complexification of a vector space X is $X^{\mathbb{C}} := X \oplus iX$;
- In the phase-space picture, this is like going from position/momentum operators to creation/annihilation operators:

$$a = \frac{q - ip}{2} \quad \text{and} \quad a^{\dagger} = \frac{q + ip}{2}$$

- Gaussian states on a phase space (X, ω) are positive-definite Lagrangian subspaces of $(X^{\mathbb{C}}, \omega^{\mathbb{C}})$;
- We can therefore define a theory of positive-semidefinite complex affine Lagrangian relations, and this captures Gaussian quantum mechanics (with infinitely-squeezed states).

Gaussian quantum mechanics, diagrammatically





Future work

- Can we give a semantics of $\mathsf{AffLagRel}_{\mathbb{R}}$ in some kind of *topological* extension of Hilbert spaces? This is necessary to reintroduce scalars.
- Can we give a satisfying description of taking infinite-squeezing limits?
- Can we extend this presentation to include non-Gaussianity? For instance, can we add Fock states and get a diagrammatic axiomatisation for Fock space? What about GKP states?
- Is there a relational semantics for the Clifford qubit ZX-calculus?

Advertisements

- Visit our website: `https://zxcalculus.com`
- Connect to our Discord (via the website)
- Come to our weekly meetings

Thanks for listening!