Graphical Symplectic Algebra

ZX seminar, 7 February 2024

University of BRISTOL

THE UNIVERSITY of EDINBURGH informatics

Relations

Relations

- a relation $X \to Y$ is a subset of $2^{X \times Y}$;

- we can compose a pair of relations $R : X \to Y$ and $T : Y \to Z$:

- there are identities: *relations form a category*
- important example: functions can be thought of as relations
- monoidal product: cartesian product

Linear & affine relations

Let $\mathbb{K}$ be a field, and X, Y, Z $\mathbb{K}$-linear spaces.

- a linear/affine relation $X \to Y$ is a linear/affine *subspace* of $X \oplus Y$;
- we can compose a pair of linear/affine relations $R : X \to Y$ and $T : Y \to Z$:
- there are identities: *linear/affine relations form a category*
- important example: linear/affine maps can be thought of as linear/affine relations
- monoidal product: direct sum
- compact structure

Symplectic linear algebra

- symplectic form: $\omega : X \oplus X \to \mathbb{R}$ that is:
  - bilinear;
  - alternating: $\omega(x, x) = 0$;
  - non-degenerate: $\omega(x, y) = 0$ for all x implies $y = 0$;
- symplectomorphisms $S : X \to Y$ s.t. $\omega_Y(Sx, Sy) = \omega_X(x, y)$
- subspaces:
  - *isotropic* if $S \subseteq S^\omega$ (so that for all $\vec{s}, \vec{t} \in S$, $\omega(\vec{t}, \vec{s}) = 0$);
  - *coisotropic* if $S^\omega \subseteq S$;
  - Lagrangian if it is both isotropic and coisotropic (so that $S = S^\omega$).

Geometric interpretation

Affine Lagrangian relations

- an affine Lagrangian relation $X \to Y$ is a affine Lagrangian *subspace* of $X \oplus Y$;
- we can compose a pair of affine Lagrangian relations $R : X \to Y$ and $T : Y \to Z$:
- there are identities: *affine Lagrangian relations form a category*
- important example: affine symplectomorphisms can be thought of as affine Lagrangian relations
- monoidal product: direct sum
- compact structure: also inherited from linear relations

Examples

By varying the field F, we can obtain semantics for a fairly wide range of subject areas:

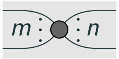
- classical mechanics;
- electrical circuits;
- (stabiliser) quantum mechanics;

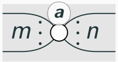
Diagrammatics

Spiders: only connectivity matters!

Graphical affine algebra: generators

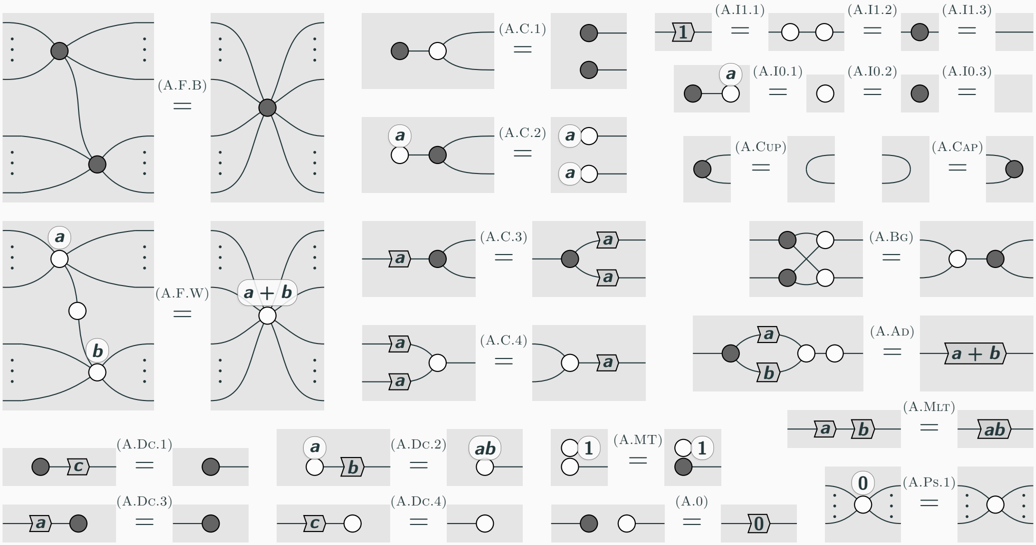
Let $\mathrm{GAA}_{\mathbb{K}}$ be the diagrammatic language generated by

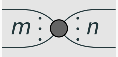




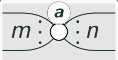
where $a \in \mathbb{K}$ and $m, n \in \mathbb{N}$, subject to the following equations...

Graphical affine algebra: equations





$$\left[\!\left[\ \cdot\ \right]\!\right]^{\mathrm{AR}}_{\mathrm{GAA}} := \left\{ \left(\begin{bmatrix} a\\ \vdots\\ a\end{bmatrix}, \begin{bmatrix} a\\ \vdots \\ a\end{bmatrix}\right) \in \mathbb{K}^n \oplus \mathbb{K}^m \;\middle|\; a\in\mathbb{K}\right\}$$



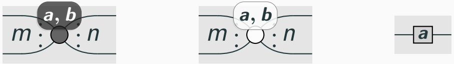
$$\left[\!\left[\ \cdot\ \right]\!\right]^{\mathrm{AR}}_{\mathrm{GAA}} := \left\{ (\vec{b}, \vec{c}) \in \mathbb{K}^n \oplus \mathbb{K}^m \;\middle|\; \sum_{j=0}^{n-1} b_j + \sum_{k=0}^{m-1} c_k = a \right\}$$

$$\left[\!\left[\ \cdot\ \right]\!\right]^{\mathrm{AR}}_{\mathrm{GAA}} := \{(b, ab) \mid b \in \mathbb{K}\}$$

Theorem

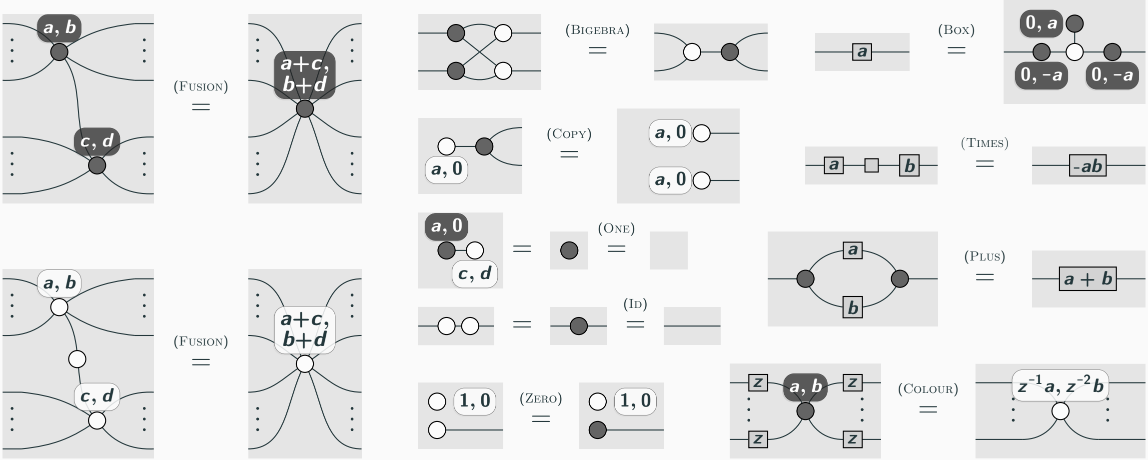
This is an equivalence of categories $\mathsf{GAA}_{\mathbb{K}} \simeq \mathsf{AffRel}_{\mathbb{K}}$.

Let $\mathrm{GSA}_{\mathbb{K}}$ be the diagrammatic language generated by

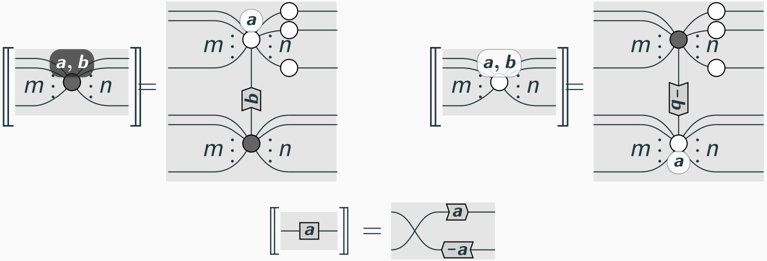


where $a, b \in \mathbb{K}$ and $m, n \in \mathbb{N}$, subject to the equations...

Graphical symplectic algebra: equations



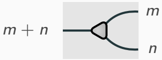
Graphical symplectic algebra: interpretation

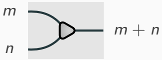


Matrices in $\mathsf{GAA}_{\mathbb{K}}$

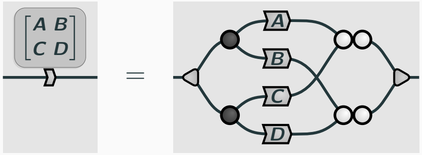
The scalable notation

We add the following syntactic sugar:



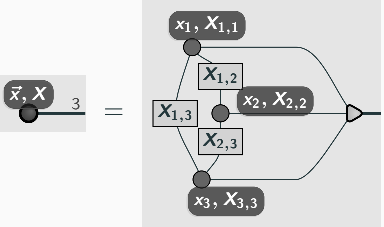


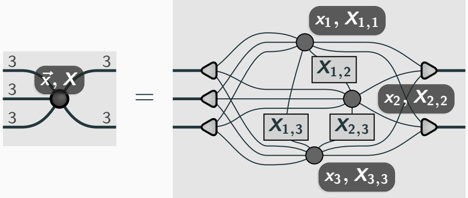
so that we can define matrix arrows,



Scalable symplectic spiders

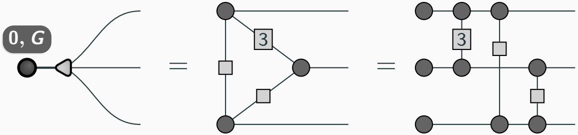
For $\vec{x} \in \mathbb{K}^n$ and *symmetric* $X \in \mathbb{K}^{n\times n}$:



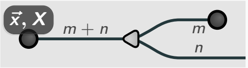


Graph states and graph-like diagrams

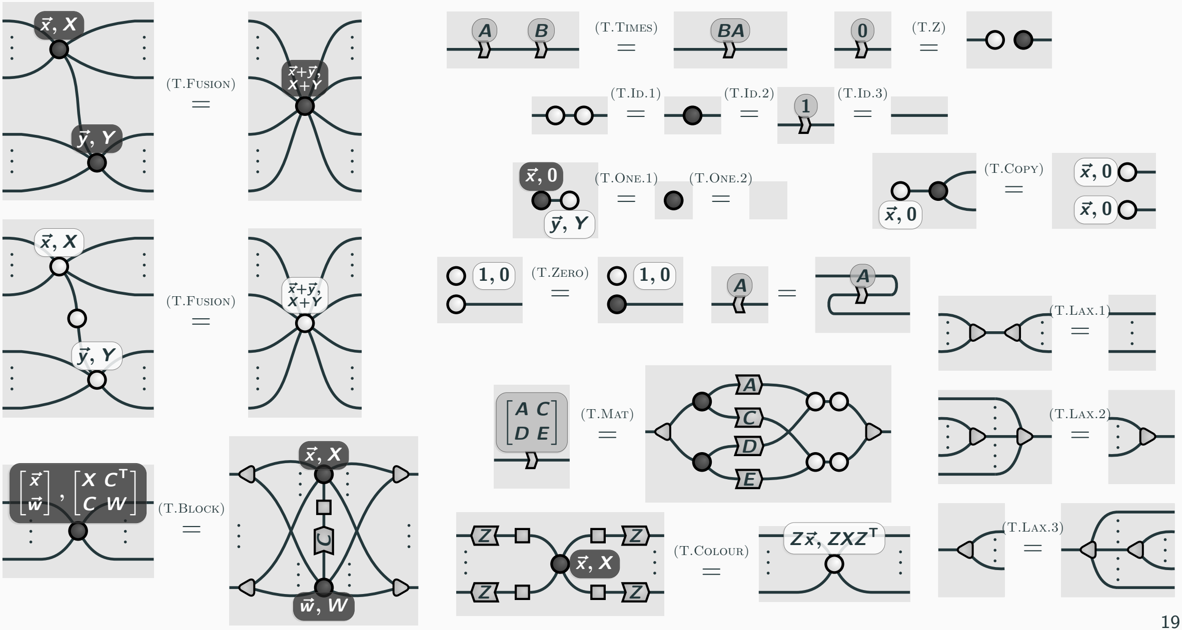
Let G be the adjacency matrix of a graph, then



Graph-like diagrams can therefore be compactly represented as:

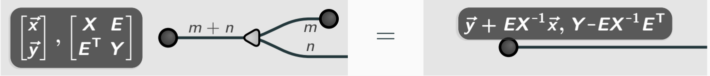


Scalable axiomatisation



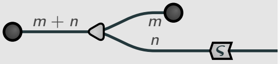
Proposition

For any *invertible* $X \in \mathrm{Sym}_m(\mathbb{K})$, $Y \in \mathrm{Sym}_n(\mathbb{K})$, $E \in \mathrm{Mat}_{\mathbb{K}}(n, m)$, $\vec{x} \in \mathbb{K}^m$, and $\vec{y} \in \mathbb{K}^n$:



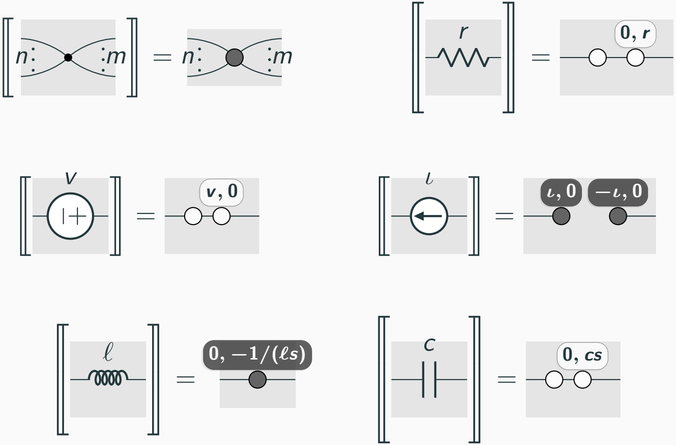
Normalisation procedure

$$\begin{bmatrix} \vec{x} \\ 0_m \\ \vec{s} \end{bmatrix}, \begin{bmatrix} 0 & 1_m & F \\ 1_m & 0 & 0 \\ F^\top & 0 & S \end{bmatrix}$$

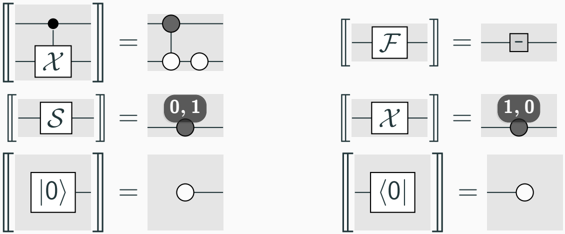


Applications

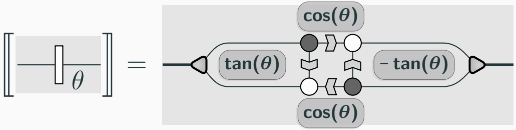
Electronic circuits

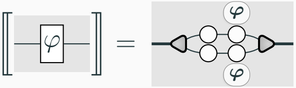


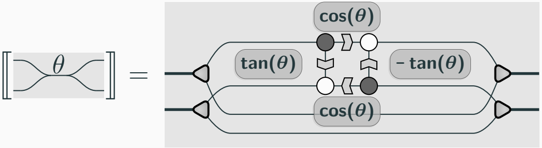
Stabiliser quantum mechanics

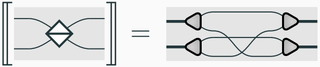


Quantum optics









Thanks for listening!