Graphical reasoning for Gaussian quantum mechanics

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Stabiliser theory in odd prime dimensions

States

Definition
Pauli groups generated by:

$$X|k\rangle = |k+1\rangle \quad \text{and} \quad Z|k\rangle = e^{i\frac{2\pi}{p}k}|k\rangle$$

Definition
Each maximal Abelian subgroup S uniquely determines a pure quantum state satisfying the property that for all $U \in S$, $U|S\rangle = |S\rangle$. We say that S is the **stabiliser group** associated to the **stabiliser state** $|S\rangle$ and vice-versa.

Definition
The **Clifford group** $\mathcal{C}_p^n$ for n quopits is the normaliser of $\mathcal{P}_p^{\otimes n}$ in the group of unitaries on $\mathcal{H}_p^{\otimes n}$: $U \in \mathcal{C}_p^n$ if and only if $UPU^\dagger \in \mathcal{P}_p^{\otimes n}$ for all $P \in \mathcal{P}_p^{\otimes n}$.

If S is a stabiliser group, and $C \in \mathcal{C}_p^n$, then $CSC^\dagger$ is the stabiliser group of the stabiliser state $\left|CSC^\dagger\right\rangle = C\left|S\right\rangle$.

Effects

Lemma
The postselected projection of a stabiliser state onto a stabiliser state is a stabiliser state.

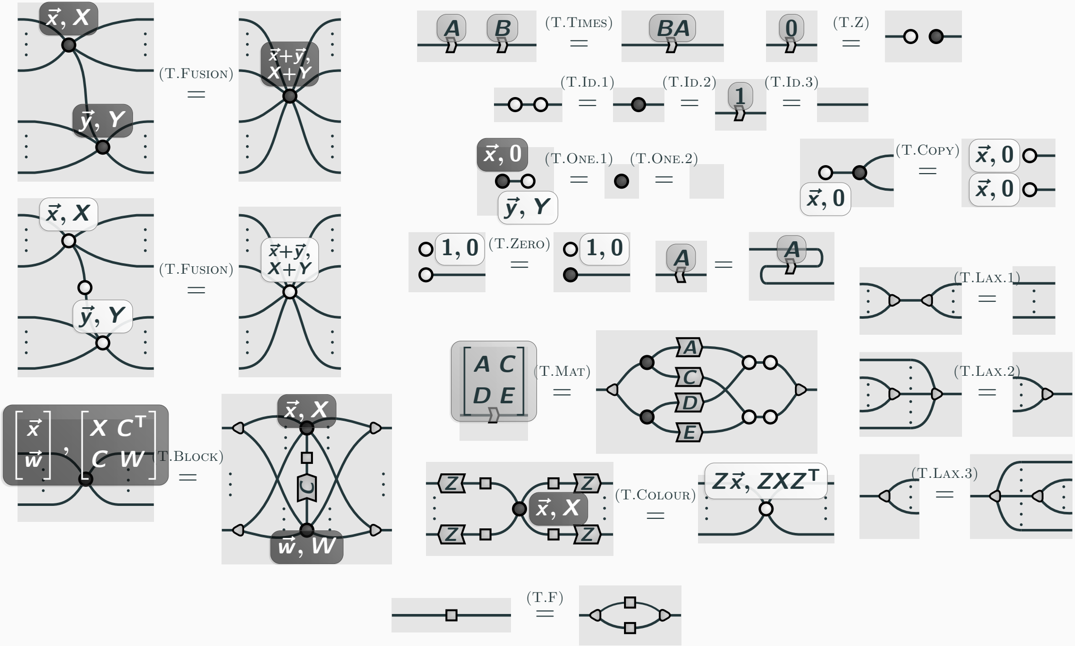
Putting it all together

Definition
The **prop of affine Lagrangian relations** $\mathsf{AffLagRel}_{\mathbb{K}}$ has:

- **Arrows** are given by affine Lagrangian relations.
- **Composition** is given by relational composition.
- **The monoidal structure** is given by the direct sum.

Theorem
There is a projective equivalence between the stabiliser theory and $\mathsf{AffLagRel}_{\mathbb{F}_p}$

The full picture



Gaussian quantum mechanics

Continuous-variable quantum computing

The state space of a quantum particle in free 1D space is the Hilbert space of complex square-integrable functions:

$$L^2(\mathbb{R}) := \left\{ \varphi : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} |\varphi(x)|^2 \, \mathrm{d}x < \infty \right\}$$

It comes equipped with **position** and **momentum** operators:

$$\hat{q}f(x) = xf(x) \quad \text{and} \quad \hat{p}f = \frac{\partial f}{\partial x}$$

and **displacement** operators:

$$\hat{X}(s)\varphi(x) := \varphi(x - s) \quad \text{and} \quad \hat{Z}(t)\varphi(x) := e^{i2\pi tx}\varphi(x)$$

Definition
An n-qumode **Gaussian state** $\varphi \in L^2(\mathbb{R}^n)$ has the form:

$$\varphi(\vec{x}) = \frac{1}{N} \exp\left(i\vec{s}^{\mathsf{T}}\vec{x}\right) \exp\left(i(\vec{x}-\vec{t})^{\mathsf{T}}\Phi(\vec{x}-\vec{t})/2\right)$$

Definition
Gaussian unitaries are those unitaries that "preserve Gaussianity."

Lemma
Post-selections onto Gaussian states preserve Gaussianity.

The CV statbiliser theory

The CV stabiliser theory is insufficient: displacements have no eigenvectors in $L^2(\mathbb{R})$.

However, physicists regularly informally employ these stabiliser “states” in calculations. For instance, a Dirac delta δ is such that

$$Z(x)\delta = \delta$$

Nullifiers

Gaussian states have **nullifiers**:

$$\hat{K}\varphi = 0$$

For instance, the vacuum state is nullified by the **annihilation** operator $\hat{a} = \hat{q} - i\hat{p}$.

Formal stabiliser states can also be understood in this picture: if φ were a state such that

$$\hat{q}\varphi = 0 \quad \text{then} \quad \hat{Z}(s)\varphi = \varphi$$

[gu˙quantum˙2009]

Nullifier theory: states

In general, the nullifiers of a Gaussian state will take the form $\sum_k a_k \hat{q}_k + b_k \hat{p}_k + c_k$ where $a_k, b_k, c_k \in \mathbb{C}$.

Definition
A complex Lagrangian relation $S : n \to m$ is **positive** when for all $\vec{v} \in S$, then $i\omega_{n,m}\left(\overline{\vec{v}}, \vec{v}\right) \geqslant 0$;

Definition
Each positive affine Lagrangian subspace L of $(\mathbb{C}^{2n}, \omega_n)$ uniquely determines a Gaussian state φ satisfying the property that for all $K \in L$, $\hat{K}\varphi = 0$. We say that S is the **nullifier space** associated to the Gaussian state φ and vice-versa.

Nullifier theory: unitary transformations

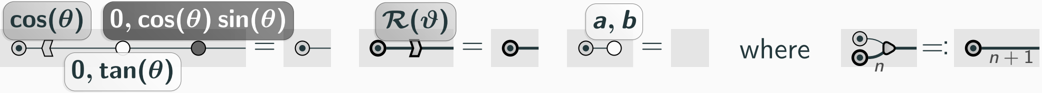
Nullifier theory: putting it all together

Definition
Let $\mathsf{AffLagRel}_{\mathbb{C}}^{+}$ denote the sub-prop of $\mathsf{AffLagRel}_{\mathbb{C}}$ of *positive* affine Lagrangian relations.

Theorem
$\mathsf{AffLagRel}_{\mathbb{C}}^{+}$ *captures the nullifier theory for Gaussian quantum mechanics, with infinitely-squeezed states.*

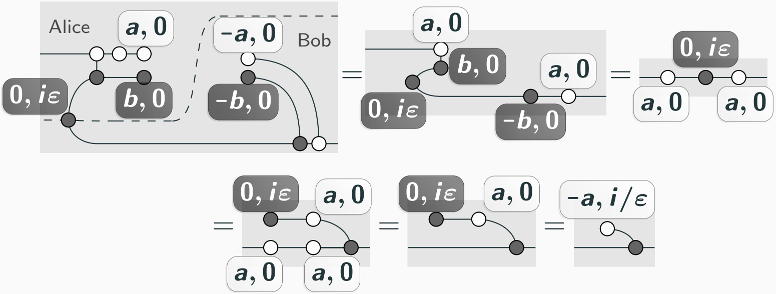
Nullifier theory: the full picture

We extend the graphical language for $\mathsf{AffLagRel}_{\mathbb{R}}$ by a single state  and the following equations:

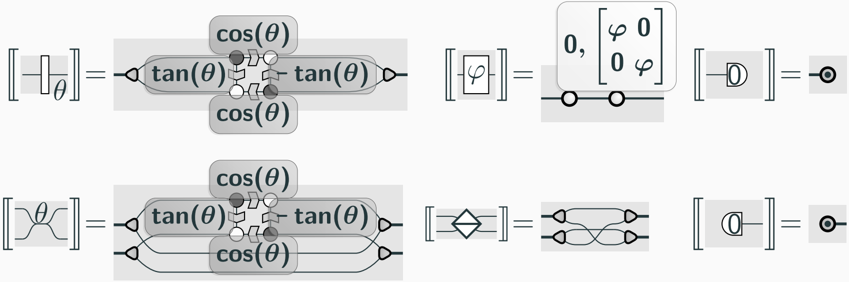


In fact, it's sound to make the identification  $=$  and use the equations of $\mathsf{AffLagRel}_{\mathbb{C}}$.

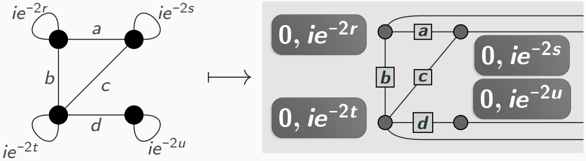
Continuous-variable quantum teleportation



Interpreting the LOv-calculus

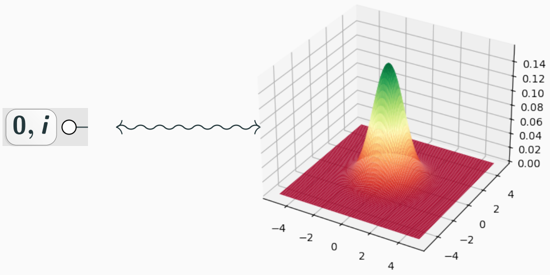


Interpreting Menicucci's graph-theoretic representaiton for Gaussian states



The phase space picture

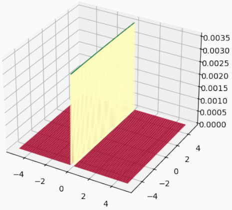
Wigner transform



Convolution in phase space

Dirac delta "distribution"

Gaussian density function



convolution by

$$\mathcal{N}\left(\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)$$