

Graphical reasoning for Gaussian quantum mechanics

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Stabiliser theory in odd prime dimensions

Definition

Pauli groups generated by:

$$X |k\rangle = |k+1\rangle \quad \text{and} \quad Z |k\rangle = e^{i\frac{2\pi}{p}k} |k\rangle$$

Definition

Each maximal Abelian subgroup S uniquely determines a pure quantum state satisfying the property that for all $U \in S$, $U|S\rangle = |S\rangle$. We say that S is the **stabiliser group** associated to the **stabiliser state** $|S\rangle$ and vice-versa.

Stabiliser groups are Lagrangian

Definition

The **Clifford group** $\mathcal{C}_p^n$ for n quopits is the normaliser of $\mathcal{P}_p^{\otimes n}$ in the group of unitaries on $\mathcal{H}_p^{\otimes n}$: $U \in \mathcal{C}_p^n$ if and only if $UPU^\dagger \in \mathcal{P}_p^{\otimes n}$ for all $P \in \mathcal{P}_p^{\otimes n}$.

If S is a stabiliser group, and $C \in \mathcal{C}_p^n$, then $CSC^\dagger$ is the stabiliser group of the stabiliser state $|CSC^\dagger\rangle = C|S\rangle$.

Lemma

The postselected projection of a stabiliser state onto a stabiliser state is a stabiliser state.

Definition

The **prop of affine Lagrangian relations** $\text{AffLagRel}_{\mathbb{K}}$ has:

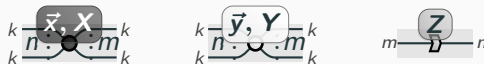
- **Arrows** are given by affine Lagrangian relations.
- **Composition** is given by relational composition.
- **The monoidal structure** is given by the direct sum.

Theorem

There is a projective equivalence between the stabiliser theory and $\text{AffLagRel}_{\mathbb{F}_p}$

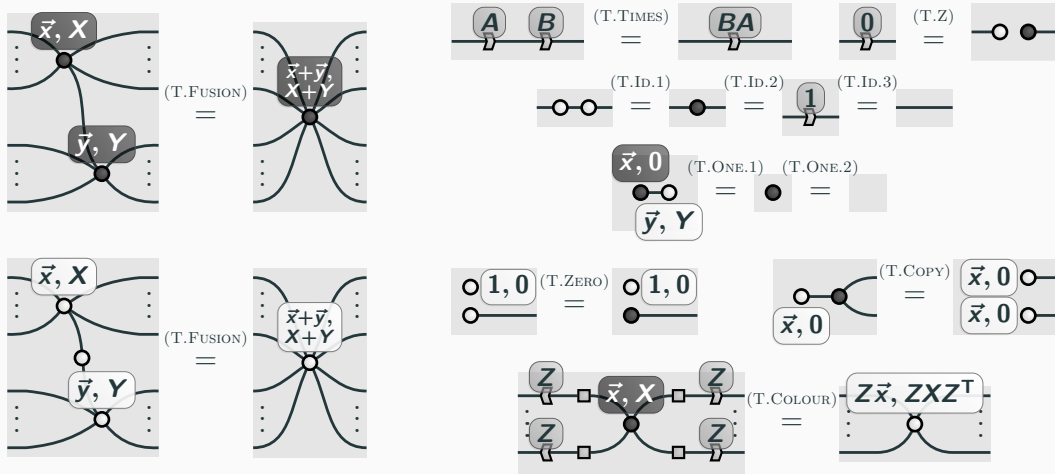
Graphical symplectic algebra: generators

$\text{GSA}_{\mathbb{K}}$ is generated by:



Robert I. Booth, Titouan Carette, and Cole Comfort. **Graphical Symplectic Algebra**. Jan. 30, 2024.

Graphical symplectic algebra: equations



Theorem. $\text{GSA}_{\mathbb{K}}$ is equivalent to the category of $\mathbb{K}$ -affine Lagrangian relations.

Gaussian quantum mechanics

Continuous-variable quantum computing

The state space of a quantum particle in free 1D space is the Hilbert space of complex square-integrable functions:

$$L^2(\mathbb{R}) := \left\{ \varphi : \mathbb{R} \rightarrow \mathbb{C} \mid \int_{\mathbb{R}} |\varphi(x)|^2 dx < \infty \right\}$$

It comes equipped with **position** and **momentum** operators:

$$\hat{q}f(x) := xf(x) \quad \text{and} \quad \hat{p}f := i\frac{\partial f}{\partial x}$$

and **displacement** operators:

$$\hat{X}(s)\varphi(x) := \varphi(x - s) \quad \text{and} \quad \hat{Z}(t)\varphi(x) := e^{i2\pi tx}\varphi(x)$$

Definition

An n -qumode **Gaussian state** $\varphi \in L^2(\mathbb{R}^n)$ has the form:

$$\varphi(\vec{x}) = \frac{1}{N} \exp\left(i\vec{s}^T \vec{x}\right) \exp\left(i(\vec{x} - \vec{t})^T \Phi(\vec{x} - \vec{t})/2\right)$$

Definition

Gaussian unitaries are those unitaries that “preserve Gaussianity.”

Lemma

Post-selections onto Gaussian states preserve Gaussianity.

The CV stabiliser theory is insufficient: displacements have no eigenvectors in $L^2(\mathbb{R})$.

However, physicists regularly informally employ these stabiliser “states” in calculations. For instance, a Dirac delta δ is such that

$$\hat{Z}(x)\delta = \delta$$

Gaussian states have **nullifiers**:

$$\hat{K}\varphi = 0$$

For instance, the vacuum state is nullified by the **annihilation** operator $\hat{a} = \hat{q} - i\hat{p}$.

Formal stabiliser states can also be understood in this picture: if φ were a state such that

$$\hat{q}\varphi = 0 \quad \text{then} \quad \hat{Z}(s)\varphi = \varphi$$

In general, $\hat{K} = \sum_k a_k \hat{q}_k + b_k \hat{p}_k$ where $a_k, b_k \in \mathbb{C}$.

Definition

A complex Lagrangian relation $S : n \rightarrow m$ is **positive** when for all $\vec{v} \in S$,
 $i\omega_{n,m}(\vec{v}, \vec{v}) \geq 0$.

Definition

Each positive affine Lagrangian subspace L of $(\mathbb{C}^{2n}, \omega_n)$ uniquely determines a Gaussian state φ satisfying the property that for all $K \in L$, $\hat{K}\varphi = 0$. We say that S is the **nullifier space** associated to the Gaussian state φ and vice-versa.

Nullifier theory: unitary transformations

Definition

Let $\text{AffLagRel}_{\mathbb{C}}^{+}$ denote the sub-prop of $\text{AffLagRel}_{\mathbb{C}}$ of *positive* affine Lagrangian relations.

Theorem

$\text{AffLagRel}_{\mathbb{C}}^{+}$ captures the nullifier theory for Gaussian quantum mechanics, with infinitely-squeezed states.

Nullifier theory: the full picture

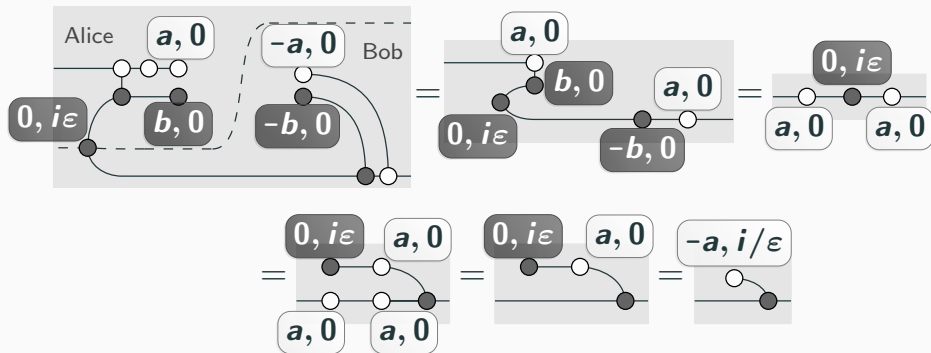
We extend the graphical language for $\text{AffLagRel}_{\mathbb{R}}$ by a single state $\odot$ and the following equations:

Diagrammatic equations for the extended $\text{AffLagRel}_{\mathbb{R}}$ graphical language:

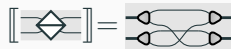
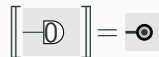
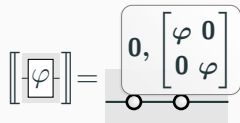
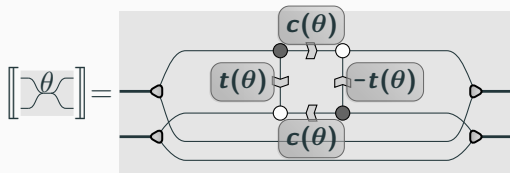
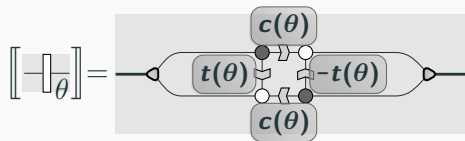
- $\text{cos}(\theta)$ (loop) $= \odot$
- $0, \text{cos}(\theta) \sin(\theta)$ (two inputs) $= \odot$
- $0, \tan(\theta)$ (two inputs) $= \odot$
- $\mathcal{R}(\vartheta)$ (loop) $= \odot$
- a, b (two inputs) $= \odot$
- where $\odot_n$ (two inputs, loop) $=: \odot_{n+1}$

In fact, it's sound to make the identification $\odot = 0, i$ and use the equations of $\text{AffLagRel}_{\mathbb{C}}$.

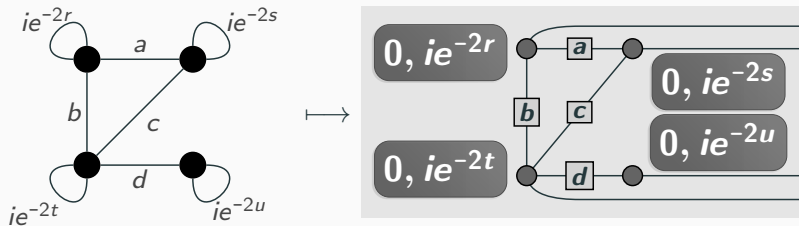
Continuous-variable quantum teleportation



Interpreting the LOv-calculus

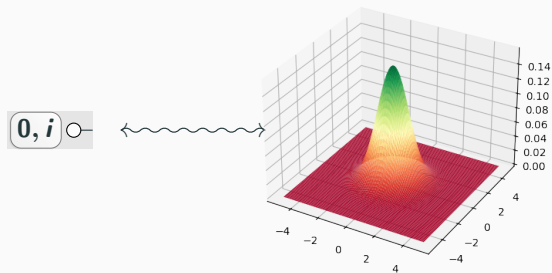


Interpreting Menicucci's graph-theoretic representation for Gaussian states



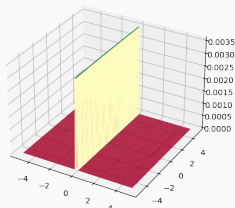
The phase space picture

Wigner transform



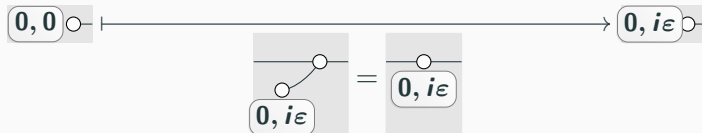
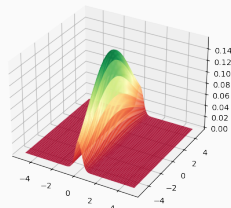
Convolution in phase space

Dirac delta “distribution”



convolution by
 $\mathcal{N}\left(\begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$

Gaussian density function



The end!