CSS surgery: implementing any CNOT in any CSS code

Robert I. Booth

Joint work with: Clément Poirson, Joschka Roffe

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What's the deal with error correction?

Binary encodings have no tolerance to errors:

$$\mathrm{bin}(42) = 00101010 \rightsquigarrow 00100010 = \mathrm{bin}(34)$$

How do we make our encodings more fault-tolerant?

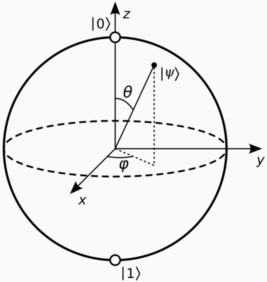
Build redundancy into encodings!

$$00101010 \quad \rightsquigarrow \quad \begin{matrix} 00101010 \\ 00101010 \\ 00101010 \end{matrix}$$

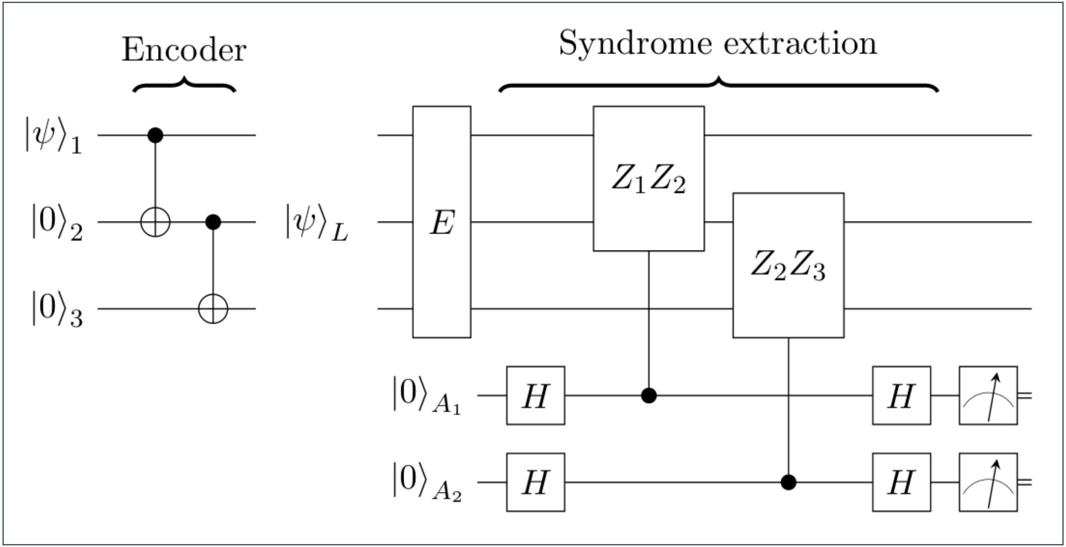
The difficulty with quantum error correction

Several factors make quantum error correction more difficult in principle:

- the no-cloning theorem
- wave-function collapse
- the continuous nature of quantum errors



The quantum repetition code



Pauli error bases

Definition

The Pauli group is generated by

$$1 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad Y := \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Theorem

Any unitary error process can be decomposed as

$$E = \alpha_I I + \alpha_X X + \alpha_Y Y + \alpha_Z Z.$$

for some complex coefficients.

Stabiliser codes

What is a stabiliser code?

Definition

A stabiliser code $\mathcal{S}$ is a subgroup of $\mathcal{P}^{\otimes n}$ that doesn't contain -1.

Definition

A state $|\varphi\rangle$ is stabilised by $\mathcal{S}$ if $S\,|\varphi\rangle = |\varphi\rangle$ for all $S \in \mathcal{S}$.

The binary representation of Pauli operators

Because of the equation $Y = iZX$, we can rewrite any Pauli as

$$P = i^{\lambda} \prod_k X^{x_k} Z^{z_k} \quad \text{where} \quad \lambda \in \mathbb{Z}/4\mathbb{Z} \quad \text{and} \quad x_k, z_k \in \mathbb{F}_2$$

Ignoring the phase, we can therefore parametrise a Pauli operator as

$$\begin{aligned} \mathbb{F}_2^n \oplus \mathbb{F}_2^n &\longrightarrow \mathcal{P}^{\otimes n}/\langle i1 \rangle \\ (\vec{x} \mid \vec{z}) &\longmapsto \prod_k X^{x_k} Z^{z_k} \end{aligned}$$

Proposition

The Paulis represented by $(\vec{x} \mid \vec{z})$ and $(\vec{x'} \mid \vec{z'})$ commute if and only if $\vec{x} \cdot \vec{z'} - \vec{z} \cdot \vec{x'} = 0$.

Parity-check matrices

Stabiliser groups can be specified by a set of **generators**:

$$\mathcal{S} = \langle X_1 X_2 X_3, Z_1 Z_2, Z_2 Z_3, Z_1 Z_3 \rangle \quad \leftrightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

What is a CSS code?

Definition

A CSS code is a stabiliser code that admits generators that are either products of Pauli X gates, or product of Pauli Z gates.

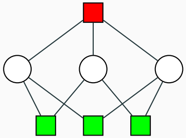
In other words, the parity-check matrix of a CSS code must take the form

$$H = \left[\begin{array}{c|c} P_X & 0 \\ 0 & P_Z \end{array}\right].$$

Tanner graphs

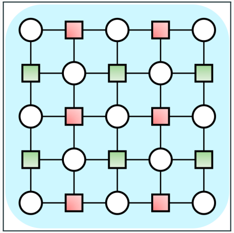
CSS codes have the nice property of being representable as certain simple graphs:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \quad \leftrightarrow$$



These are called Tanner graphs.

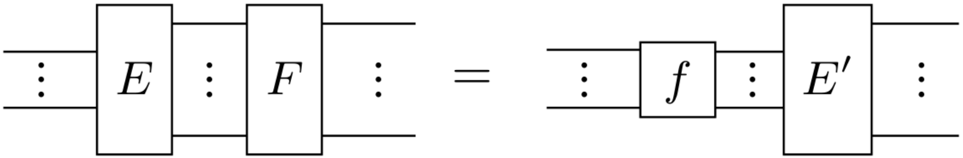
There is famous CSS code, the surface code, which is given by the Tanner graph:



CSS surgery

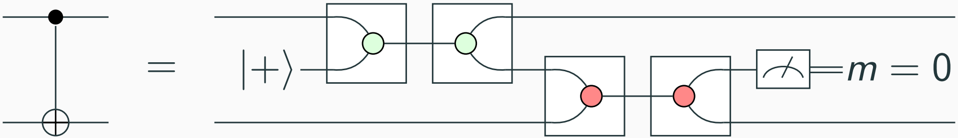
Logical and physical operations, diagrammatically

Implementing a logical operation f for an encoder E, means finding an F such that:

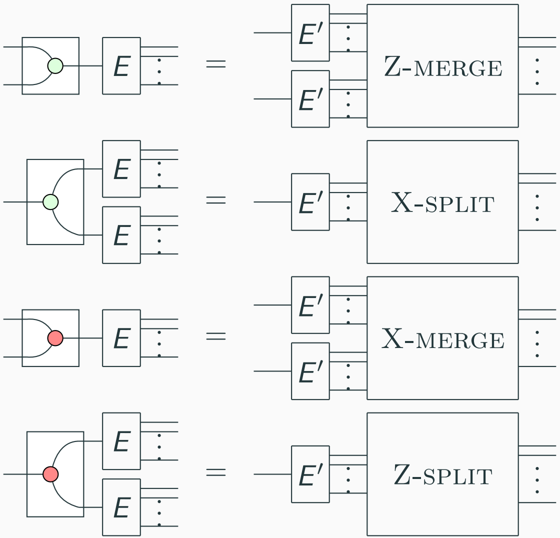


(often, $E = E'$, but this isn't compulsory)

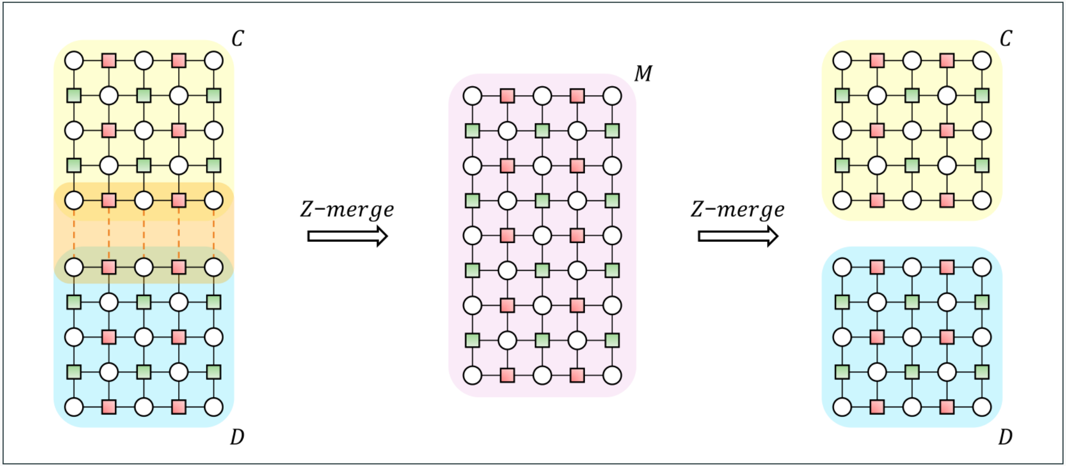
Decomposition of the CNOT for lattice surgery



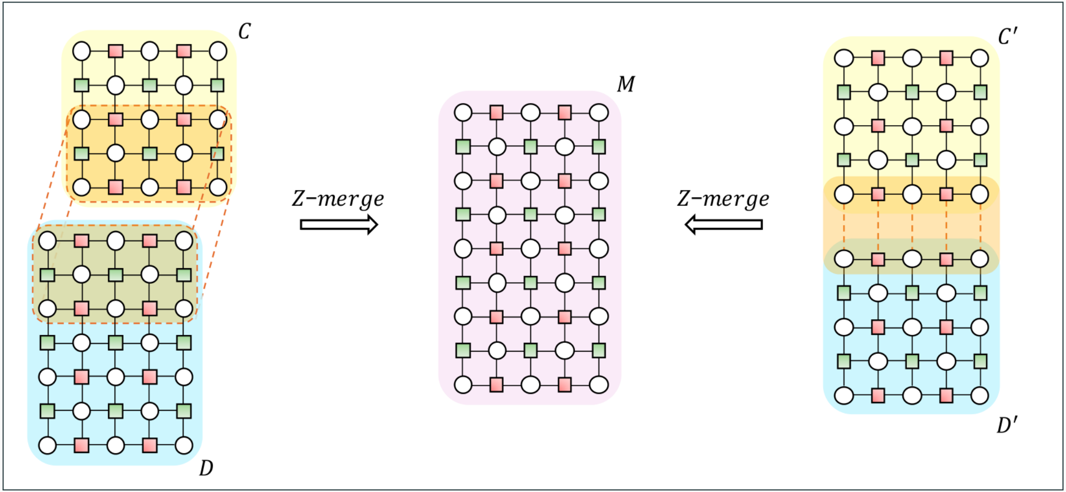
Merging and splitting



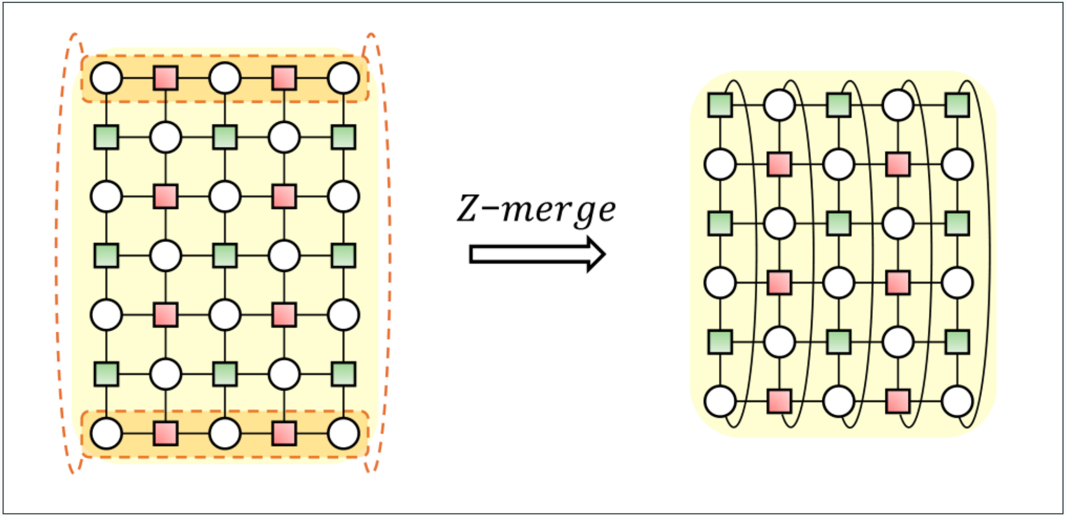
Merging at the level of Tanner graphs



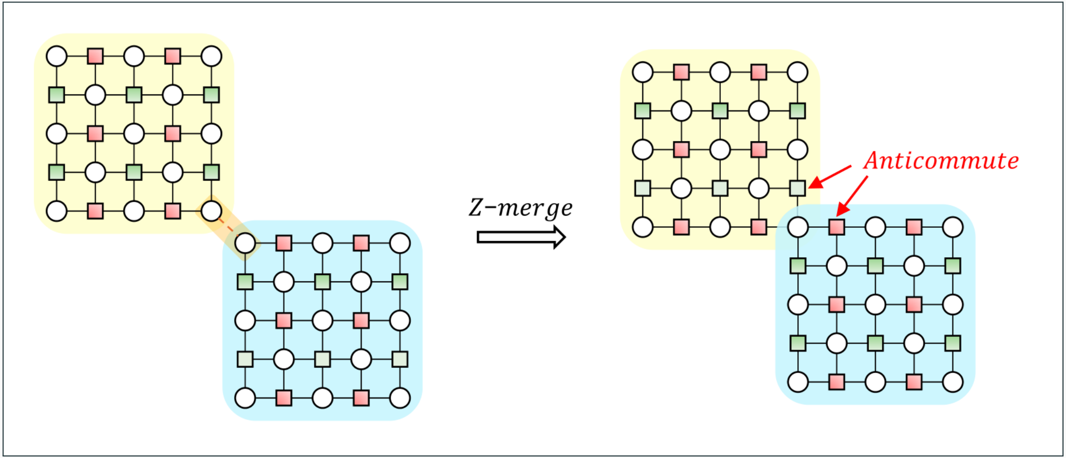
The difficulties with CSS surgery



The difficulties with CSS surgery



The difficulties with CSS surgery



Chain complexes

The problem is in the choice of representation of CSS codes:

- Tanner graphs (= parity-check matrices) forget the linear structure of the stabiliser group
- stabiliser groups track the linear structure, but no the operational meaning of Tanner graphs