The nullifier theory of Gaussian quantum states

Robert I. Booth

QAT, 27 November 2024

The stabiliser theory

States

Pauli groups generated by:

$$X|k\rangle = |k+1\rangle \quad \text{and} \quad Z|k\rangle = e^{i\frac{2\pi}{p}k}|k\rangle$$

States

Pauli groups generated by:

$$X|k\rangle = |k+1\rangle \quad \text{and} \quad Z|k\rangle = e^{i\frac{2\pi}{p}k}|k\rangle$$

Definition

Each maximal Abelian subgroup $\mathcal{S}$ uniquely determines a pure quantum state satisfying the property that for all $U \in \mathcal{S}$, $U|\mathcal{S}\rangle = |\mathcal{S}\rangle$. We say that $\mathcal{S}$ is the **stabiliser group** associated to the **stabiliser state** $|\mathcal{S}\rangle$ and vice-versa.

States

Pauli groups generated by:

$$X|k\rangle = |k+1\rangle \quad \text{and} \quad Z|k\rangle = e^{i\frac{2\pi}{p}k}|k\rangle$$

Definition

Each maximal Abelian subgroup $\mathcal{S}$ uniquely determines a pure quantum state satisfying the property that for all $U \in \mathcal{S}$, $U|\mathcal{S}\rangle = |\mathcal{S}\rangle$. We say that $\mathcal{S}$ is the **stabiliser group** associated to the **stabiliser state** $|\mathcal{S}\rangle$ and vice-versa.

This state is given by

$$|\mathcal{S}\rangle\langle\mathcal{S}| = \prod_{U \in \mathcal{S}} \frac{1+U}{2}$$

Definition

The Clifford group $\mathcal{C}_p^n$ for n quopits is the normaliser of $\mathcal{P}_p^{\otimes n}$ in the group of unitaries on $\mathcal{H}_p^{\otimes n}$: $U \in \mathcal{C}_p^n$ if and only if $UPU^\dagger \in \mathcal{P}_p^{\otimes n}$ for all $P \in \mathcal{P}_p^{\otimes n}$.

Unitary transformations

Definition

The Clifford group $\mathcal{C}_p^n$ for n quopits is the normaliser of $\mathcal{P}_p^{\otimes n}$ in the group of unitaries on $\mathcal{H}_p^{\otimes n}$: $U \in \mathcal{C}_p^n$ if and only if $UPU^\dagger \in \mathcal{P}_p^{\otimes n}$ for all $P \in \mathcal{P}_p^{\otimes n}$.

If $\mathcal{S}$ is a stabiliser group, and $C \in \mathcal{C}_p^n$, then $C\mathcal{S}C^\dagger$ is the stabiliser group of the stabiliser state $\left|C\mathcal{S}C^\dagger\right\rangle = C\left|\mathcal{S}\right\rangle$.

Gaussian quantum mechanics

Continuous-variable quantum computing

The state space of a quantum particle in free 1D space is the Hilbert space of complex square-integrable functions:

$$L^2(\mathbb{R}) := \left\{ \varphi : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} |\varphi(x)|^2 \, \mathrm{d}x < \infty \right\}$$

Continuous-variable quantum computing

The state space of a quantum particle in free 1D space is the Hilbert space of complex square-integrable functions:

$$L^2(\mathbb{R}) := \left\{ \varphi : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} |\varphi(x)|^2 \,\mathrm{d}x < \infty \right\}$$

It comes equipped with **position** and **momentum** operators:

$$\hat{q}f(x) = xf(x) \quad \text{and} \quad \hat{p}f = i\frac{\partial f}{\partial x}$$

Continuous-variable quantum computing

The state space of a quantum particle in free 1D space is the Hilbert space of complex square-integrable functions:

$$L^2(\mathbb{R}) := \left\{\varphi : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} |\varphi(x)|^2 \,\mathrm{d}x < \infty\right\}$$

It comes equipped with **position** and **momentum** operators:

$$\hat{q}f(x) = xf(x) \quad \text{and} \quad \hat{p}f = i\frac{\partial f}{\partial x}$$

and **displacement** operators:

$$\hat{X}(s)\varphi(x) := \varphi(x - s) \quad \text{and} \quad \hat{Z}(t)\varphi(x) := e^{i2\pi tx}\varphi(x)$$

The CV stabiliser theory

The CV stabiliser theory is inadequate: displacements have no eigenvectors in $L^2(\mathbb{R})$.

The CV stabiliser theory

The CV stabiliser theory is inadequate: displacements have no eigenvectors in $L^2(\mathbb{R})$.

However, physicists regularly informally employ these stabiliser "states" in calculations. For instance, a Dirac delta δ is such that

$$Z(x)\delta = \delta$$

Gaussian quantum mechanics

Definition

Gaussian quantum states are those states in $L^2(\mathbb{R}^n)$ whose quadrature measurement statistics are Gaussian.

Definition

Gaussian quantum states are those states in $L^2(\mathbb{R}^n)$ whose quadrature measurement statistics are Gaussian.

An n-qumode Gaussian state $\varphi \in L^2(\mathbb{R}^n)$ has the form:

$$\varphi(\vec{x}) = \frac{1}{N} \exp\left(i\vec{s}^{\mathsf{T}}\vec{x}\right) \exp\left(i(\vec{x}-\vec{t})^{\mathsf{T}}\Phi(\vec{x}-\vec{t})/2\right)$$

Gaussian quantum mechanics

Definition

Gaussian quantum states are those states in $L^2(\mathbb{R}^n)$ whose quadrature measurement statistics are Gaussian.

An n-qumode Gaussian state $\varphi \in L^2(\mathbb{R}^n)$ has the form:

$$\varphi(\vec{x}) = \frac{1}{N} \exp\left(i\vec{s}^{\mathsf{T}}\vec{x}\right) \exp\left(i(\vec{x} - \vec{t})^{\mathsf{T}} \Phi (\vec{x} - \vec{t})/2\right)$$

Definition

Gaussian transformations are those linear maps on $L^2(\mathbb{R}^n)$ that "preserve Gaussianity."

First and second-order moments

Theorem

A Gaussian state ψ is completely determined by its first and second moments:

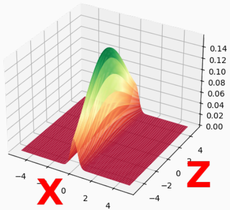
1. the displacement vector:
$$d_i := \langle \hat{x}_i \rangle_\psi$$
2. the covariance matrix:
$$\Sigma_{i,j} := \frac{1}{2} \langle \{\Delta\hat{x}_i, \Delta\hat{x}_j\} \rangle_\psi \quad \text{where} \quad \Delta\hat{x}_i = \hat{x}_i - \langle x_i \rangle_\psi$$

where $\Sigma + i\Omega > 0$.

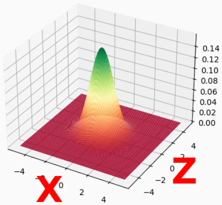
There are "special" Gaussian states $\varnothing_n$, whose moments are $\mathbf{d} = \mathbf{0}_{2n}$ and $\Sigma = \frac{1}{n} 1_{2n}$.

Gaussian quantum mechanics in phase-space

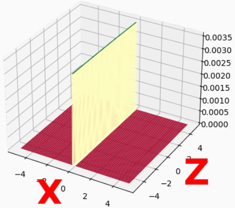
Gaussian state



Vacuum state

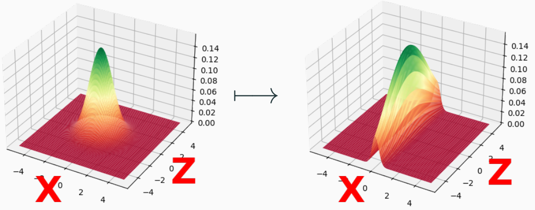


Dirac delta distribution

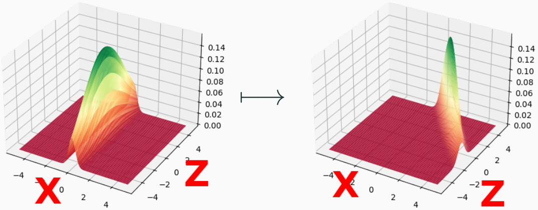


Other Gaussians from the vacuum

We can squeeze Gaussian distributions:



as well as displace and rotate them:



Gaussian unitaries are symplectomorphisms

Theorem

Gaussian unitaries act on Gaussian states as

$$\mathbf{d} \longmapsto S\mathbf{d}$$
$$\Sigma \longmapsto S\Sigma S^{\mathsf{T}}$$

for some symplectic matrix S.

But...

What is the Gaussian equivalent of a stabiliser group? And how does this appear in the symplectic picture?

The nullifier theory

Ladder operators

The annihilation and creation operators are:

$$\hat{a} := \frac{\hat{q} - i\hat{p}}{2} \quad \text{and} \quad \hat{a}^\dagger := \frac{\hat{q} + i\hat{p}}{2}$$

Ladder operators

The annihilation and creation operators are:

$$\hat{a} := \frac{\hat{q} - i\hat{p}}{2} \quad \text{and} \quad \hat{a}^\dagger := \frac{\hat{q} + i\hat{p}}{2}$$

The **ladder operators** take the form

$$\hat{a}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \sum_k (\alpha_k a_k + \beta_k a_k^\dagger) \quad \text{for} \quad \alpha, \beta \in \mathbb{C}^n.$$

Ladder operators

The annihilation and creation operators are:

$$\hat{a} := \frac{\hat{q} - i\hat{p}}{2} \quad \text{and} \quad \hat{a}^\dagger := \frac{\hat{q} + i\hat{p}}{2}$$

The **ladder operators** take the form

$$\hat{a}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \sum_k (\alpha_k a_k + \beta_k a_k^\dagger) \quad \text{for} \quad \alpha, \beta \in \mathbb{C}^n.$$

In other words, the ladder operators live in the **complexification** of the phase space $(\mathbb{C}^{2n}, \omega_n^{\mathbb{C}})$, such that $[\hat{a}(x), \hat{a}(y)] = i\omega^{\mathbb{C}}(x, y)$.

Nullifiers

Gaussian states have **nullifiers**:

$$\hat{a}(x)\varphi = 0 \quad \text{where} \quad x \in \mathbb{C}^{2n}.$$

Nullifiers

Gaussian states have **nullifiers**:

$$\hat{a}(x)\varphi = 0 \quad \text{where} \quad x \in \mathbb{C}^{2n}.$$

For instance, the vacuum state is nullified by the **annihilation** operator $\hat{a} = \hat{q} - i\hat{p}$.

Nullifiers

Gaussian states have **nullifiers**:

$$\hat{a}(x)\varphi = 0 \quad \text{where} \quad x \in \mathbb{C}^{2n}.$$

For instance, the vacuum state is nullified by the **annihilation** operator $\hat{a} = \hat{q} - i\hat{p}$.

Formal stabiliser states can also be understood in this picture: if δ were a state such that

$$\hat{q}\delta = 0 \quad \text{then} \quad \hat{Z}(s)\delta = \delta$$

Nullifiers

Gaussian states have **nullifiers**:

$$\hat{a}(x)\varphi = 0 \quad \text{where} \quad x \in \mathbb{C}^{2n}.$$

For instance, the vacuum state is nullified by the **annihilation** operator $\hat{a} = \hat{q} - i\hat{p}$.

Formal stabiliser states can also be understood in this picture: if δ were a state such that

$$\hat{q}\delta = 0 \quad \text{then} \quad \hat{Z}(s)\delta = \delta$$

What is the connection between the nullifiers and the moments of a Gaussian state?

Intermezzo: complex structures

Definition

A **complex structure** on a real vector space is a matrix J such that $J^2 = -1$.

Definition

A complex structure on a real symplectic space is ω-**compatible** if it is symplectic and furthermore if $\omega(x, Jx) > 0$ for all non-zero x.

From covariances to complex structures and back

Where do nullifiers live?

The complexification of a vector space V is $V^{\mathbb{C}} := \mathbb{C} \otimes V \cong V \oplus iV$.

A complex structure J on V *complexifies* to a complex structure $J^{\mathbb{C}}$ on $V^{\mathbb{C}}$. It is now diagonalisable and has eigenvalues $\pm i$, so that $V^{\mathbb{C}} \cong V_J^+ \oplus V_J^-$.

Nullifier spaces

Definition

A complex Lagrangian subspace $S \subseteq \mathbb{C}^{2n}$ is **strictly negative** when for all $\mathbf{v} \in S$, $i\omega_n(\overline{\mathbf{v}}, \mathbf{v}) < 0$.

Proposition

$S \subseteq \mathbb{C}^{2n}$ is strictly negative if and only if there is an ω_n-compatible complex structure J such that S is the $(-i)$-eigenspace of $J^{\mathbb{C}}$.

What about infinitely squeezed states?

Proposition

Lagrangian subspaces of $\mathbb{R}^{2n}$ are in bijection with complex Lagrangian subspaces L of $\mathbb{C}^{2n}$ such that $i\omega(\bar{\mathbf{x}}, \mathbf{y}) = 0$ for all $(\mathbf{x}, \mathbf{y}) \in L$.

In other words, the physicist's Gaussian states are perfectly captured by the negative (affine) Lagrangian subspaces of $\mathbb{C}^{2n}$, i.e. nullifier spaces.

Gaussian unitaries in the nullifier theory

Describing Gaussian processes

Unitary transformations in the stabiliser theory

Definition

The Clifford group $\mathcal{C}_p^n$ for n quopits is the normaliser of $\mathcal{P}_p^{\otimes n}$ in the group of unitaries on $\mathcal{H}_p^{\otimes n}$: $U \in \mathcal{C}_p^n$ if and only if $UPU^\dagger \in \mathcal{P}_p^{\otimes n}$ for all $P \in \mathcal{P}_p^{\otimes n}$.

If $\mathcal{S}$ is a stabiliser group, and $C \in \mathcal{C}_p^n$, then $C\mathcal{S}C^\dagger$ is the stabiliser group of the stabiliser state $\left|C\mathcal{S}C^\dagger\right\rangle = C\left|\mathcal{S}\right\rangle$.

Lemma

The postselected projection of a stabiliser state onto a stabiliser state is a stabiliser state.

Namely, for stabiliser groups $\mathcal{S}$ and $\mathcal{T}$, the state $(\langle\mathcal{T}| \otimes I) |\mathcal{S}\rangle$ is a stabiliser state, i.e. the map

$$|\mathcal{S}\rangle \longmapsto (\langle\mathcal{T}| \otimes I) |\mathcal{S}\rangle$$

is a stabiliser operation.

Relational composition

Definition

A **relation** $R : X \to Y$ is just a subset of $X \times Y$.

Putting it all together

Definition

The stabiliser theory has:

- **Processes** are given by affine Lagrangian relations.
- **Sequential composition** is given by relational composition.
- **Parallel composition** is given by the direct sum.

Nullifier theory: putting it all together

Definition

The nullifier theory has:

- **processes** given by nullifier subspaces of $\mathbb{C}^{2m+2n}$, viewed as relations $\mathbb{C}^{2m} \to \mathbb{C}^{2n}$;
- **sequential composition** given by relational composition;
- **parallel composition** given by the direct sum.

To be continued...

- "Graphical Symplectic Algebra", Robert I. Booth, Titouan Carette, Cole Comfort
- "Complete equational theories for classical and quantum Gaussian processes", Robert I. Booth, Titouan Carette, Cole Comfort
- "CSS surgery: compiling any CNOT in any CSS code", Clément Poirsson, Joschka Roffe, and Robert I. Booth
- "Picturing stabiliser codes", Robert I. Booth and Cole Comfort

Some questions for you

- Should I write a paper on this?
- Is it too late to switch to calling these "annihilators" instead of "nullifiers"?
- Is there a formula to go from the nullifier space to the (projection onto) a Gaussian state?
- Is there any hope of nullifiers generalising beyond Gaussians? Is there such a thing as a nullifier of a non-Gaussian state?